NAME: ____________________

SCHOOL: ____________________

University of Houston
High School Contest – Spring 2008 Calculus Test

1. $\lim_{x \to 0} \sec\left(\frac{\sin 2\pi x}{6x}\right) =$

   (a) 1
   (b) $2/\sqrt{3}$
   (c) 2
   (d) $\sqrt{2}$
   (e) The limit does not exist

2. Suppose $f(0) = 2$, $f'(0) = 3$, $f(2) = -1$, $f'(2) = -2$, $g(0) = 2$, $g'(0) = 4$, $g(2) = -1$, $g'(2) = 0$. If $h(x) = f(g(x))$, then $h'(0) =$.

   (a) -4
   (b) -8
   (c) 0
   (d) 12
   (e) 8

3. Let f be some function for which you know only that

   $$\text{if} \qquad 0 < |x - 3| < 1, \qquad \text{then} \qquad |f(x) - 4| < 0.1.$$

   Which of the following statements are necessarily true?

   I. If $|x - 3| < 0.1$, then $|f(x) - 4| < 0.01$.

   II. If $|x - 2.6| < 0.3$, then $|f(x) - 4| < 0.1$.

   III. If $0 < |x - 3| < 0.5$, then $|f(x) - 4| < 0.1$.

   IV. $\lim_{x \to 3} f(x) = 4$.

   (a) II and IV
   (b) IV only
   (c) I and II
   (d) II and III
   (e) II, III and IV

4. A function f is defined on an interval $[a, b]$. Which of the following statements could be false?

   I. If $f(a)$ and $f(b)$, have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   II. If f is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   III. If f is continuous on $[a, b]$ and there is a point c in (a, b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

   IV. If f has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

   (a) II only
   (b) II, III
   (c) I, III, IV
   (d) II, IV
   (e) I, III

5. An equation for the normal line to the curve $2x^3 + 2y^2 = 5xy$ at the point $(1, 2)$ is

   (a) $3x - 4y = 2$
   (b) $3x + 4y = -5$
   (c) $3x + 4y = 11$
   (d) $4x - 3y = -2$
   (e) $4x + 3y = 10$

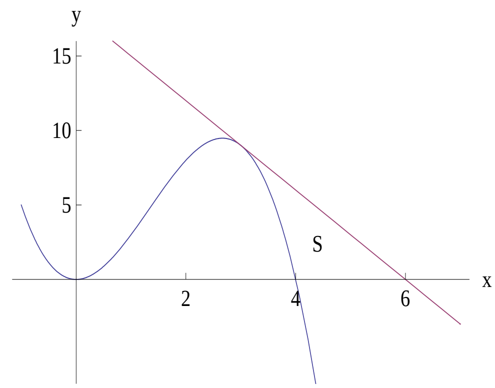
6. Set

$$g(x) = \begin{cases} a\sqrt{x+1} & 0 < x < 3, \\ bx + 2 & 3 \le x < 5. \end{cases}$$

   The values of a and b such that g is differentiable on $(0, 5)$ are:

   (a) $a = 4,\ b = 2$
   (b) $a = -8/5,\ b = -2/5$
   (c) $a = 2,\ b = 3/2$
   (d) $a = -1,\ b = -3/4$
   (e) $a = 8/5,\ b = 2/5$

7. Set $f(x) = 4x^2 - x^3$, and let $\mathcal{L}$ be the line $y = 18 - 3x$, where $\mathcal{L}$ is tangent to the graph of f. Let S be the region bounded by the graph of f, the line $\mathcal{L}$ and the x-axis. The area of S is:



(a) 103/12
(b) 43/6
(c) 101/12
(d) 95/12
(e) 53/6

8. A particle is moving along the x-axis so that its velocity at time t, $0 \leq t \leq 10$ is

$$v(t) = \ln\left(t^2 - 3t + 3\right).$$

During which time intervals is the particle moving to the left?

(a) $2 < t \leq 10$
(b) The particle never moves left
(c) $1 < t < 2$
(d) $0 \leq t < 1$
(e) $0 < t < 5$

9. Let R be the region bounded by the graph of $f(x) = \sqrt{x-1}$, the vertical line $x = 10$, and the x-axis. Find the volume of the solid generated when R is revolved about the line $y = 3$.

(a) $\dfrac{135\pi}{2}$
(b) $\dfrac{99\pi}{2}$
(c) $\dfrac{189\pi}{2}$
(d) $\dfrac{119\pi}{2}$
(e) $\dfrac{137\pi}{2}$

10. If $\int_1^4 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = 7$, and $\int_1^8 f(x)\,dx = 11$, then $\int_8^3 f(x)\,dx =$.

(a) -9
(b) 13
(c) -1
(d) 9
(e) -13

11. If f is a continuous function and $F(x) = \int_0^x \left[(2t+3)\int_t^2 f(u)\,du\right] dt$, then $F''(2) =$

(a) $-2f(2)$
(b) $-7f(2)$
(c) $7f'(2)$
(d) $3f'(2)$
(e) $7f(2)$

12. A curve in the plane is defined by the parametric equations: $x = e^{2t} + 2e^{-t}$, $y = e^{2t} + e^{t}$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $7x + 10y = -8$
(b) $5x - 6y = -11$
(c) $5x - 3y = 7$
(d) $10x - 7y = 8$
(e) $3x - 2y = 3$

13. Which of the following series converges to 2?

I. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ II. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ III. $\sum_{n=1}^{\infty} \frac{4}{3^n}$

(a) I only
(b) II only
(c) I and III
(d) III only
(e) II and III

14. The function $f(x) = \int_3^x \sqrt{16+t^2}\,dt$ has an inverse. Find $\left(f^{-1}\right)'(0)$.

(a) 5
(b) 1/4
(c) 4
(d) 3
(e) 1/5

15. Suppose that the power series $\sum_{k=0}^{\infty} a_k(x-1)^k$ converges at $x = 3$. Which of the following series must be convergent?

I. $\sum_{k=0}^{\infty} a_k$

II. $\sum_{k=0}^{\infty} a_k 3^k$

III. $\sum_{k=0}^{\infty} (-1)^k a_k$

IV. $\sum_{k=0}^{\infty} (-1)^k a_k 2^k$

(a) I only
(b) II and III
(c) I and III
(d) II, III, and IV
(e) III and IV

16. A point (x, y) is moving along a curve $y = f(x)$. At the instant when the slope of the curve is $-3/5$, the x-coordinate of the point is decreasing at the rate of 4 units per second. The rate of change, in units per second, of the y-coordinate is

(a) $12/5$

(b) $3/20$

(c) $-12/5$

(d) $5/12$

(e) $-20/3$

17. The region in the first quadrant bounded by the graph of $y = e^{2x}$, the vertical line $x = \ln 3$, and the x-axis is revolved about the y-axis. Find the volume of the solid that is generated.

(a) $2\pi\,[3\ln 3 - 2]$

(b) $\pi\,[9\ln 3 + 4]$

(c) $9\pi/2$

(d) $\pi\,[9\ln 3 - 4]$

(e) $\pi\left[\frac{5}{2}\ln 3 - \frac{1}{2}\right]$

18. The length of the graph of $f(x) = \ln \sec x,\ 0 \le x \le \pi/3$ is:

(a) $3 + \sqrt{2}$

(b) $\ln\left(2 + \sqrt{3}\right)$

(c) $\ln\left(\sqrt{3}\right)$

(d) $2 + \sqrt{3}$

(e) $\ln\left(\frac{1 + \sqrt{3}}{2}\right)$

19. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are necessarily true?

I. If $a_n > 0$ for all n and $a_n \to L$, then $L > 0$.

II. If $\{a_n\}$ is not bounded below, then it diverges.

III. If $a_n \geq 0$ for all n and $a_n \to L$, then $L \geq 0$.

IV. If $\{a_n\}$ is increasing and bounded above, then it converges.

(a) III only
(b) I, III
(c) II, IV
(d) II, III, IV
(e) II, III

20. Let f be a function such that $|f^{(n)}(x)| \leq 1$ for all x and n. Find the least integer n such that the Taylor polynomial of degree n at $x = 0$ approximates $f(1/2)$ to within 0.0005.

(a) 2
(b) 3
(c) 4
(d) 5
(e) 6

21. Evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$.

(a) $\pi/2$
(b) 2
(c) 1
(d) $\pi/4$
(e) divergent

22. The curves $r = \sqrt{2}$ and $r = 2\cos\theta$ are shown in the figure. Find the area of the shaded region.

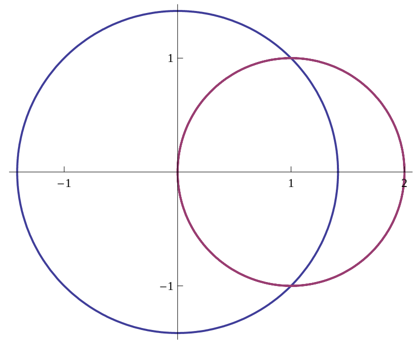
(a) $\pi + 1$

(b) $\dfrac{\pi + 1}{2}$

(c) $\pi - \frac{1}{4}$

(d) $\dfrac{\pi - 2}{4}$

(e) $\dfrac{\pi - 1}{2}$



23. The function f is infinitely differentiable, $f(2) = 4$, and

$$f^{(n)}(2) = \frac{(n-1)!}{3^n} \quad \text{for all } n \geq 1.$$

The interval of convergence of the Taylor series for f in powers of $x - 2$ is:

(a) $-3 < x < 3$

(b) $-1 \leq x < 5$

(c) $-3 \leq x \leq 3$

(d) $-1 < x < 5$

(e) $-1 < x \leq 5$

24. $\displaystyle\lim_{x\to\infty} e^{-x^2} \int_0^x 2te^{t^2}\, dt =$

(a) 0

(b) 1/2

(c) 1

(d) e

(e) The limit does not exist.

25. An advertising company has designed a campaign to introduce a new product to city of 2 million people. Let $P = P(t)$ denote the number of people who are aware of the product at time t and assume that P increases at a rate proportional to the number of people still unaware of the product. If no one knew about the product at the beginning of the campaign $[P(0) = 0]$ and 40% of the people are aware of the product after 3 months of advertising, how long will it take for 90% of the population to be aware of the product?

(a) 11.65 mos

(b) 10.23 mos

(c) 14.72 mos

(d) 12.39 mos

(e) 13.52 mos