NAME____________________________ SCHOOL______________________________

**University of Houston**
**High School Math Contest – 2016**
**Algebra II Test**

1. Let *a* and *b* be the *x*-coordinates of the points where the function $f\left(x\right)=\dfrac{2x^3+x^2}{x^3+x^2-2x+1}$ intersects its horizontal asymptote. Find the value of $ab^2+a^2b$.

   A. -6
   B. -8
   C. 8
   D. 6
   E. 0

2. Let *S* be the set of all integers that are in the domain of the function:
$$f\left(x\right)=\frac{\sqrt{2-\ln(x+2)}}{x^2+4x-12}.$$
   How many subsets does *S* have?

   A. 4
   B. 8
   C. 32
   D. 64
   E. 128

3. All of the students taking a certain test are either 14 or 15 years old. 80% of the students are boys. 25% of all girls are 14 years old. If the number of 15 year olds is triple the number of 14 year olds, what percent of the students are boys who are 14 years old?

   A. 35
   B. 20
   C. 15
   D. 10
   E. 5

4. Given: $\ln(2) = a$, $\ln(6) = b$, and $\ln(0.1) = c$.

Express $\ln(112{,}500)$ in terms of *a, b* and *c*.

A. $2a + 2b + 5c$
B. $2b + 5c - 5a$
C. $3a + 2b - 5c$
D. $2a + 2b - 5c$
E. $2b - 5c - 5a$

5. Let $f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ \dfrac{8x+2}{x+1}, & \text{if } x \geq 0 \end{cases}$, and $g(x) = \dfrac{10}{x-1}$. Find $(f^{-1} \circ g)(3)$.

A. 3
B. 17
C. 7
D. 1
E. 3/2

6. Simplify: $$\frac{x^3 - x^2 - 2x}{x^2 + x - 2} \cdot \left(\frac{x^3 - 1}{x^2 - 2x}\right)^2 \div \frac{x^4 + x^3 - x - 1}{x^2 - 4}$$

A. $x + \dfrac{1}{x} + 1$
B. $x - 1 + \dfrac{1}{x}$
C. $x + 2$
D. $\dfrac{x^2 - x + 1}{x^2}$
E. $\dfrac{x^2 - 1}{x^2}$

7. Find the remainder when $3^{2015}$ is divided by 7.

A. 1
B. 2
C. 3
D. 4
E. 5

8. Which of the following is a factor of the polynomial: $P(x) = x^4 + 1$?

A. $x+1$
B. $x^2+\sqrt{2}x-1$
C. $x^2-\sqrt{2}x+1$
D. $x^2-1$
E. $x^2+1$

9. Simplify the expression:

$$\left(\frac{4^{2015}-1}{2^{2016}-2}\right)\cdot\sqrt{\frac{4^{2000}+4^{1999}}{4^{2000}-4^{1998}}}\cdot\left(\frac{3^{385.5}}{3^{1+4+9+16+\ldots+10^2}}\right)$$

A. $3\left(2^{2015}-1\right)$
B. $\sqrt{3}\left(2^{2015}-1\right)$
C. $\sqrt{3}\left(2^{2015}+1\right)$
D. $2^{2015}+1$
E. $6\left(2^{2015}+1\right)$

10. Let $i=\sqrt{-1}$. Which of the following is equivalent to $i^{2015}\left((1+i)^{2014}+(1-i)^{2013}\right)$ ?

A. $2^{1006}i$
B. $2^{1006}(1+i)$
C. $2^{1006}(1-i)$
D. $2^{1006}(-1+i)$
E. $2^{1007}(1+i)$

11. A group of 10 people (6 mathematicians and 4 chemists) will be seated at a conference. Let ***N*** be the number of ways they can be seated at a row of 10 chairs given that all mathematicians must sit side by side. Let ***M*** be the number of ways they can be seated around a round table given that all chemists must sit side by side. Find the ratio $\dfrac{N}{M}$.

A. 5/4
B. 5
C. 20
D. 20/7
E. 5/7

12. Let $s_n$ be a sequence defined recursively as

$$s_1=1,\quad s_2=1,\quad \text{and}\quad s_n=\frac{s_{n-1}\cdot s_{n-2}+s_{n-1}^2}{s_{n-2}}\quad \text{for } n>2.$$

Find the value of $\dfrac{s_{21}}{s_{19}}$.

A. 220
B. 440
C. 420
D. 210
E. 380

13. Consider the system of equations:

$$\log_{10}(1000xy)-(\log_{10}x)(\log_{10}y)=4$$
$$\log_{10}\left(\frac{x}{100}\right)-\log_{10}(10y^3)=1$$

If $(x,y)$ is a solution of this system, then which of the following is one of the possible values of $x^2\cdot y^4$?

A. $10^2$
B. $10^8$
C. $10^{18}$
D. $10^4$
E. $10^{-4}$

14. Let $6ab$ be a three digit number that is divisible by 12. If $a+b>a$, then find the sum of all possible two digit numbers $ab$.

A. 432
B. 468
C. 288
D. 384
E. 372

15. A box contains 4 sets of cards numbered from 1 to 5 (four 1s, four 2s, …, four 5s). Sam chose 2 cards from the box and did not return them. Given that Sam chose two matching cards (that is, both cards had the same number), now Tom will chose 2 cards from the box. What is the probability that Tom's cards are matching?

A. 25/153
B. 21/143
C. 15/141
D. 17/149
E. 21/164

16. Let $z$ be a complex number and $\overline{z}$ be its conjugate. Given: $\dfrac{\overline{z}}{z+60} = 4+2i$, which of the following is the imaginary part of $z$?

A. -240/19
B. 120/19
C. 480/17
D. 240/17
E. 83/21

17. Let $A$ be the remainder when the number

$$8 \times 9 \times 98 \times 99 \times 998 \times 999 \times 9998$$

is divided by 100. Find the sum of the digits of $A$.

A. 6
B. 8
C. 9
D. 10
E. 16

18. Evaluate the sum: $$\frac{4}{\log_2\left(160{,}000^5\right)} + \frac{2}{\log_5\left(160{,}000^5\right)}$$

A. 2/5
B. 1/5
C. 5/16
D. 3/10
E. 1/10

19. Consider all functions $f(x)$ with these properties: $f(x)$ is a second degree polynomial with integer coefficients and integer zeros; the remainder when $f(x-2)$ is divided by $x-2$ is 30. How many such functions are there?

A. 13
B. 14
C. 56
D. 55
E. 163

20. A pound is 16 ounces, 9 pennies weigh 1 ounce. Bob, Charlie and David each have a bag of pennies. Bob's bag weighs 3.5 pounds and Charlie's bag weighs 5 pounds. When Charlie and David's bags are combined, they have a total of $10.08. David borrowed some money from Bob to be able to double his money. What is the weight of Bob's bag after he lent the money to David?

A. 2 pounds
B. 1.5 pounds
C. 1 pound and 12 ounces
D. 1 pound and 10 ounces
E. 2 pounds and 2 ounces

21. Let $a$ and $b$ be relatively prime positive integers. For the polynomial $P(x)=(ax+b)^{200}$, the coefficients of $x^2$ and $x^3$ are equal. Find the value of $b-2a$.

A. 64
B. 112
C. 664
D. 48
E. 32

22. A digital clock displays 1 or 2 digits for the hours and 2 digits for the minutes. Between the hours of 2:00 pm and 7:00 pm, how many displays will represent a perfect square?

A. 14
B. 12
C. 9
D. 7
E. 5

23. Let R be the region containing all points satisfying the inequalities:

$$y < \frac{5x}{2} - 2$$
$$2y > x + 4$$

If $(a,b)$ is a point in R with integer coordinates, find the minimum value of $a+b$.

A. 3
B. 5
C. 7
D. 8
E. 9

24. Let $a$ and $b$ be positive odd integers such that $a < b < 100$. Given that arithmetic mean of $a$ and $b$ is exactly 2 more than geometric mean of $a$ and $b$, how many ordered pairs $(a,b)$ are there?

A. 2
B. 3
C. 4
D. 6
E. 7

25. Consider the solution set of the system:

$$x + y = 2$$
$$x^3 + y^3 = 20$$

Which of the following is the **y-coordinate** of one of the points in this set?

A. $\dfrac{1}{1-\sqrt{3}}$
B. $\dfrac{2}{1+\sqrt{3}}$
C. $-2$
D. $2\sqrt{3}$
E. $1+\sqrt{3}$

26. Let $R$ be the region containing all points satisfying:

$$y < |x+6| + |2-x|,$$
$$|x+1| \le 9,$$
$$y \ge 0.$$

Find the area of region $R$.

A. 160
B. 192
C. 216
D. 132
E. 196

27. Let $a, b, c$ be positive numbers satisfying:

$$abc = 1$$
$$a + \frac{1}{c} = 7$$
$$b + \frac{1}{a} = 3$$

Find the value of $\left(c + \frac{1}{b}\right)a$.

A. 2/5
B. 6/5
C. 4/3
D. 3/5
E. 4/5

**The following questions are part of this test, but they are not multiple choice. For the following 3 questions, write your answer on the answer sheet as a number. For example:**

**25, 0, 4.5, -2.7,** $1+2\sqrt{3}$**,** $\frac{3-4\sqrt{7}}{2}$**, 1/4 , 12/13 or 50/11**

**are acceptable answers. Show your work on the empty space below each question and write your final answer on the answer sheet.**

28. Jack and Kay are flying their drones under certain rules. Jack's drone travels at a constant speed and Kay's drone has to fly at a speed that is 20% slower than Jack's. Kay's drone takes a two minute break after every 200 meters. Jack's drone takes a one minute break after every 100 meters. They start flying the drones from the same point and at the same time. The drones follow the same route to reach a target which is 6 kilometers away. If both drones reach the target at the same time, then how many minutes does it take for the drones to reach the target?

29. A sequence of numbers $r_1, r_2, ..., r_{10}$ has the property that for every integer $k$, $1 \le k \le 10$, the number $r_k$ is $k$ less than the sum of the other 9 numbers. Find the fourth term of this sequence.

30. Given the conic sections:

$$x^2 - 4x + y^2 = 5, \qquad \frac{(x+2)^2}{9} + y^2 = 1, \qquad y^2 + 4 = 4x.$$

Consider all points where **any two** of these objects intersect; let $S$ be the set of the $x-$coordinates of all these points. Find the **sum of the smallest and the largest** elements in this set.

*(Write your final answer as a number in one of the acceptable forms explained before.)*