University of Houston
High School Math Contest – Spring 2016 Calculus Test

NAME: ______________________________

SCHOOL: ______________________________

1. $\lim_{x\to 0} \dfrac{\tan^3 2x}{x^3} =$
   (a) 1/8
   (b) 4
   (c) 8
   (d) 1/4
   (e) The limit does not exist

2. $\lim_{x\to 0} \dfrac{e^{ax^2} - \cos 2x}{x^2} = 8$. What is a?
   (a) $a = 3$
   (b) $a = 6$
   (c) $a = 4$
   (d) $a = 2$
   (e) $a = 1$

3. $\lim_{h\to 0} \dfrac{\int_{\frac{\pi}{6}}^{\frac{\pi}{6}+h} \frac{\sin x}{x}\, dx}{h} =$
   (a) 0
   (b) $\dfrac{\sqrt{3}}{2}$
   (c) $\dfrac{\pi}{3}$
   (d) $\dfrac{3}{\pi}$
   (e) $\dfrac{6}{\pi}$

4. The values of A and B such that

$$f(x) = \begin{cases} x^2 - 2, & x \leq 2 \\ Ax^2 + Bx, & x > 2 \end{cases}$$

is everywhere differentiable are:

(a) $a = 1/2,\ B = 2$

(b) $a = 1/4,\ B = 1/2$

(c) $a = -3/2,\ B = 2$

(d) $a = 2,\ B = -1$

(e) $a = 3/2,\ B = -2$

5. Set $f(x) = x^3 - 6x$. The value of c that satisfies the Mean-Value Theorem for Derivatives on the interval $[0, 5]$ for f is

(a) $\dfrac{5}{\sqrt{3}}$

(b) 0

(c) $\dfrac{1}{\sqrt{3}}$

(d) $\dfrac{5}{3}$

(e) $\dfrac{-5}{\sqrt{3}}$

6. If f and g are twice differentiable. and $h(x) = f[g(x)]$, then $h''(x) =$

(a) $f''[g(x)]g'(x) + f'[g(x)]g''(x)$

(b) $f''[g'(x)]\,[g'(x)]^2 + f[g'(x)]g''(x)$

(c) $f''[g(x)]\,[g'(x)]^2 + f'[g(x)]g''(x)$

(d) $f''[g(x)]g'(x) + f[g(x)]g''(x)$

(e) $f'[g(x)]\,[g'(x)]^2 + f[g(x)]g''(x)$

7. The normal line to the curve $y = 2x^2 - 4x + 2$ at the point $(2, 2)$ has an x-intercept at:

(a) $(5/2, 0)$
(b) $(10, 0)$
(c) $(5, 0)$
(d) $(2, 0)$
(e) $(-5/2, 0)$

8. When the local linearization of $f(x) = \sqrt{4 + \ln(1 + x)}$ near $x = 0$ is used, an estimate of $f(0.08)$ is:

(a) 1.98
(b) 2.01
(c) 2.02
(d) 2.04
(e) 2.06

9. Which of the following statements about the function $f(x) = x^4 - 4x^3$ is true?

(a) The graph of f has no points of inflection and the function has one relative extremum.
(b) The graph of f has one point of inflection and the function has two relative extrema.
(c) The graph of f has two points of inflection and the function has two relative extrema.
(d) The graph of f has two points of inflection and the function has one relative extremum.
(e) The graph of f has one point of inflection and the function has one relative extremum.

10. The position of a particle moving along the x-axis is given by

$$s(t) = \int_0^t (u^2 - 6u + 8)\, du$$

for $0 \le t \le 8$. For what values of t is the speed of the particle increasing?

(a) $2 < t < 4$ only
(b) $t > 4$ only
(c) $t > 3$ only
(d) $0 < t < 2$ and $t > 4$
(e) $2 < t < 3$ and $t > 4$

11. The function $f(x) = x^3 + 2x - 6$ has an inverse. If the graph of f passes through the point $(2, 6)$, then $(f^{-1})'(6) =$

(a) $1/14$
(b) $1/10$
(c) -6
(d) 14
(e) 10

12. A particle moves along the ellipse $3x^2 + 2y^2 - 4y = 3$ so that $dy/dt = 3$ at all times t. The speed of the particle when it is at the point $(1, 2)$ is:

(a) $\sqrt{14}$
(b) $\sqrt{13}$
(c) 5
(d) $2\sqrt{3}$
(e) 4

13. The average value of $f(x) = x \ln x$ on $[1, e]$ is:

(a) $\dfrac{e^2 + 1}{4}$
(b) $\dfrac{e^2 + 1}{4(e+1)}$
(c) $\dfrac{e + 1}{4}$
(d) $\dfrac{e^2 + 1}{4(e-1)}$
(e) $\dfrac{2e^2 + 1}{4(e-1)}$

14. Given that $\int_0^2 f(x)\,dx = \frac{8}{3}$, $\int_1^2 f(x)\,dx = \frac{4}{3}$, and $\int_0^3 f(x)\,dx = \frac{11}{3}$, find $\int_3^1 f(x)\,dx$.

(a) $-5/3$
(b) 2
(c) $7/3$
(d) $5/3$
(e) $-7/3$

15. $\int e^{2x}\sqrt{e^x - 3}\;dx =$

(a) $\frac{2}{5}(e^x - 3)^{5/2} + 2(e^x - 3)^{3/2} + C$
(b) $\frac{4}{5}(e^x - 3)^{5/2} + \frac{3}{5}(e^x - 3)^{3/2} + C$
(c) $\frac{5}{2}(e^x - 3)^{5/2} - \frac{3}{2}(e^x - 3)^{3/2} + C$
(d) $\frac{2}{5}(e^x - 3)^{5/2} + \frac{2}{3}(e^x - 3)^{3/2} + C$
(e) $\frac{4}{5}(e^x - 3)^{5/2} + (e^x - 3)^{3/2} + C$

16. The region bounded by $y = e^x$, $y = 1$, and the line $x = 2$ is rotated about the y-axis. Which of the following integrals gives the volume of the solid which is generated:

$$(A)\ \pi\int_0^2 e^{2x}\,dx, \qquad (B)\ 2\pi\int_0^2 x(e^x - 1)\,dx, \qquad (C)\ \pi\int_0^2 \left(e^{2x} - 1\right)\,dx$$

$$(D)\ 2\pi\int_1^{e^2} y(2 - \ln y)\,dy, \qquad (E)\ \pi\int_1^{e^2} \left(4 - \ln^2 y\right)\,dy$$

(a) (A) and (D)
(b) (B) and (E),
(c) (A), (D) and (E)
(d) (C) and (D)
(e) (B) only

17. If f is a continuous function and $F(x) = \int_0^x \left[(t^2 + 2t)\int_t^2 f(u)\,du\right] dt,$ then $F''(2) =$

(a) $4f(2)$
(b) $4f'(2)$
(c) $-4f'(2)$
(d) $-8f(2)$
(e) $8f(2)$

18. A cylindrical container 2 feet in diameter and 4 feet high is half full of oil weighing 50 pounds per cubic foot. The work done, in foot-pounds, in pumping the oil to an outlet one foot above the top of the container is:

(a) 100π
(b) 250π
(c) 400π
(d) 500π
(e) 800π

19. $\displaystyle\int_4^\infty \frac{2\,dt}{16+t^2} =$

(a) $\pi/2$
(b) $\pi/8$
(c) $\pi/16$
(d) $\pi/4$
(e) Divergent

20. If $x = 2\sin\theta$, then $\displaystyle\int_1^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$ is equivalent to

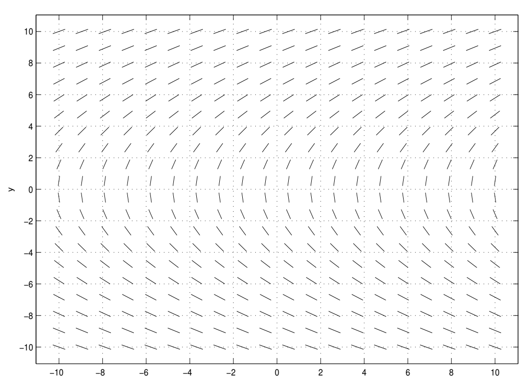
(a) $4\displaystyle\int_1^2 \sin^2\theta\,d\theta$
(b) $2\displaystyle\int_0^{\pi/2} \sin\theta\,\tan\theta\,d\theta$
(c) $2\displaystyle\int_{\pi/6}^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$
(d) $4\displaystyle\int_0^{\pi/2} \sin^2\theta\,d\theta$
(e) $4\displaystyle\int_{\pi/6}^{\pi/2} \sin^2\theta\,d\theta$

21. If $\dfrac{dy}{dx} = \dfrac{2x^3-1}{y}$ and $y = 2$ when $x = 1$, then, when $x = 2$, $y =$

(a) $\pm\sqrt{17}$
(b) $\pm\sqrt{15}$
(c) ± 5
(d) $\pm 2\sqrt{3}$
(e) $\pm 2\sqrt{5}$

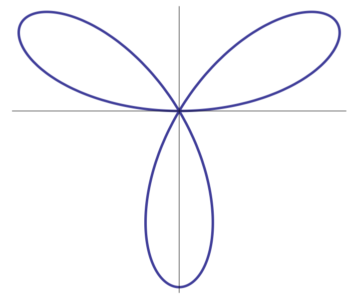
22. Which differential equation which has the slope field

(a) $\dfrac{dy}{dx} = \dfrac{3}{x}$
(b) $\dfrac{dy}{dx} = \dfrac{x}{y}$
(c) $\dfrac{dy}{dx} = 3y$
(d) $\dfrac{dy}{dx} = \dfrac{3}{y}$
(e) $\dfrac{dy}{dx} = x - y$



23. The curve $r = 2\sin 3\theta$ is shown in the figure. The area enclosed by one loop of the curve is:

(a) $\pi/6$
(b) $\pi/2$
(c) $\pi/3$
(d) π
(e) $2\pi/3$



24. A curve in the plane is defined by the parametric equations

$$x = t^2 + 1, \quad y = \frac{1}{3}t^3 - t + 2, \quad 0 \le t \le 3.$$

Find the length of the curve.

(a) $2\sqrt{3}$
(b) 12
(c) 8
(d) 10
(e) $\sqrt{14}$

25. If 50 grams of a radioactive substance decays to 25 grams in two hours, then, to the nearest gram, the amount left after 5 hours is:

(a) 5
(b) 7
(c) 9
(d) 10
(e) 12

26. If a block of ice melts at the rate of $\dfrac{64}{2t+3}$ cm^3/min, then the closest approximation to the amount of ice which melts during the first three minutes is:

(a) $29\,\text{cm}^3$
(b) $32\,\text{cm}^3$
(c) $35\,\text{cm}^3$
(d) $40\,\text{cm}^3$
(e) $48\,\text{cm}^3$

27. Which infinite series converge(s)?

(I) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$ (II) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{4n^5+2n+1}}$ (III) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$

(a) I and III
(b) I only
(c) I and II
(d) III only
(e) I, II and III

28. Suppose that the power series $\displaystyle\sum_{k=0}^{\infty} a_k(x-2)^k$ converges at $x=4$. Which of the following series must be convergent?

I. $\displaystyle\sum_{k=0}^{\infty} a_k 3^k$ II. $\displaystyle\sum_{k=0}^{\infty} a_k$ III. $\displaystyle\sum_{k=0}^{\infty} (-1)^k a_k$ IV. $\displaystyle\sum_{k=0}^{\infty} (-1)^k a_k 2^k$

(a) II only
(b) I and III
(c) II, III, and IV
(d) III and IV
(e) II and III

29. Using two terms of an appropriate Maclaurin series (Taylor series at $x = 0$), estimate $\int_0^1 \frac{1 - \cos x}{x}\, dx$.

(a) $\frac{19}{96}$

(b) $\frac{23}{96}$

(c) $\frac{1}{4}$

(d) $\frac{25}{96}$

(e) undefined; the integral is improper

30. A function f is infinitely differentiable and has the property that $\left|f^{(k)}(x)\right| \le k2^k$ for all $x \in (-1, 1)$ and all k. Find the least integer n such that the Taylor polynomial of degree n in powers of x for f approximates $f(1/4)$ to within 0.0005

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7