NAME_____ SCHOOL_____

University of Houston High School Math Contest – 2018

Pre-Calculus Test

1. $f(x)$ is a quadratic function satisfying $f(2)=17$, $f(3)=35$ and $f(4)=61$. Find the remainder when $f(x)$ is divided by $x-1$.

   A) 3
   B) 7
   C) 9
   D) 11
   E) 4
   F) None of these

2. Let M be a non-zero digit. When the sum of the 50 numbers

   $\underbrace{M}_{1 \text{ digit}}$
   $\underbrace{MM}_{2 \text{ digits}}$
   $\underbrace{MMM}_{3 \text{ digits}}$
   $\vdots$
   $\underbrace{MMMM.....M}_{50 \text{ digits}}$

   is divided by 9, the remainder is 3. Find the sum of all possible values of M.

   A) 7
   B) 10
   C) 13
   D) 15
   E) 17
   F) None of these

3. Let $f(x)=18e^{2x-1}$, $g(x)=2\log_3(10x+2)$, and $h(x)=\dfrac{4x-1}{6x+3}$.

   Find the value of $\left(f^{-1}\circ g\circ h^{-1}\right)\left(\dfrac{1}{2}\right)$.

   A) $\dfrac{1+\ln 3}{2}$
   B) $\dfrac{1-\ln 3}{2}$
   C) $\ln\left(\dfrac{1}{3}\right)-1$

D) $\ln\left(\frac{1}{3}\right)+\frac{1}{2}$

E) $1+\ln 3$

F) None of these

4. Let $f(x)$ be a real valued function defined as: $f(x)=\frac{x+1}{x^2+4}$.

Find the range of this function.

A) $\left[\frac{1-2\sqrt{5}}{8},\frac{1+2\sqrt{5}}{8}\right]$

B) $\left[\frac{-\sqrt{3}}{4},\frac{\sqrt{5}}{4}\right]$

C) $\left[\frac{-\sqrt{3}}{8},\frac{\sqrt{3}}{8}\right]$

D) $\left[\frac{1-\sqrt{3}}{8},\frac{1+\sqrt{3}}{8}\right]$

E) $\left[\frac{1-\sqrt{5}}{8},\frac{1+\sqrt{5}}{8}\right]$

F) None of these

5. Let $f(x)=\frac{\left(1+x+x^2+x^3\right)(1-x)^2}{1-x-x^2+x^3}$, where $x>1$.

Evaluate the following: $f\left(\sqrt{2}\right)+f^{-1}(10)$.

A) 4
B) 5
C) 6
D) 7
E) 10
F) None of these

6. Let a be a real number with $-10 < a < 0$. The vertical asymptote of the function $f(x) = \ln(8x+2)$ is also a vertical asymptote of $g(x) = \dfrac{x+\sin x}{2x+\cos(a\pi x)}$.

   Find the sum of all possible values of a.

   A) $-\dfrac{8}{3}$
   B) $-\dfrac{22}{3}$
   C) $-\dfrac{52}{3}$
   D) –4
   E) –8
   F) None of these

7. Let y be the product of all real solutions of the equation $x^{\ln 4} - 6 \cdot 2^{\ln x} + 8 = 0$.

   Evaluate the following expression: $\ln(y^2) + \log_y(e)$.

   A) 5/3
   B) 19/3
   C) 13/6
   D) 10/3
   E) 17/6
   F) None of these

8. Let S be the set of all real solutions of the inequality: $(x-1)^2 < |x-1| + 6$.

   Let T be the domain of the function $g(x) = \arcsin\left(\tan\left(\dfrac{x}{4} + \dfrac{\pi}{12}\right)\right)$. Find $S \cap T$.

   A) $\left(\dfrac{-4\pi}{3}, 4\right)$
   B) $\left[\dfrac{-4\pi}{3}, -2\right)$
   C) $\left(-2, \dfrac{2\pi}{3}\right]$
   D) $\left(-\dfrac{4\pi}{3}, \dfrac{2\pi}{3}\right]$
   E) $\left[-\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right]$
   F) None of these

9. Let A, B, C be sets defined as:

$$A = \{(x, x) : x \in \mathbb{R}\}$$
$$B = \{(x, 3-x) : x \in \mathbb{R}\}$$
$$C = \{(x, x+4) : x \in \mathbb{R}\}$$

If $(p, q) \in A \cap B$ and $(r, s) \in B \cap C$, find the value of $\dfrac{p-r}{q+s}$.

A) 1/3
B) 1/4
C) 3/4
D) 4/5
E) 2/5
F) None of these

10. Let n be an integer and $f(n)$ be defined as the first non-zero digit of the number n from the right. For example, $f(10234500) = 5$ and $f(123) = 3$. Find the value of $f\left(x\right)$ at $x = \sqrt{2000^{2020}}$.

A) 2
B) 4
C) 6
D) 8
E) 1
F) None of these

11. Let a, b, c, d be integers satisfying:

$$a \log_{10} 2 + b \log_{1000} 3 + c \log_{100} 5 + d \log_{0.1} 11 = 2018.$$

Evaluate $4a - b - c - 2d$.

A) 8072
B) 4036
C) 12034
D) 10090
E) 2018
F) None of these

12. Let p be a real number. Given that the distance between the foci of the ellipse $\frac{x^2}{(p+1)^2}+\frac{y^2}{(p-1)^2}=1$ is 12, find the length of the major axis of this ellipse.

A) 14
B) 16
C) 18
D) 20
E) 24
F) None of these

13. Let $x=\frac{1+\cos(40^0)}{\cos(55^0)\cdot\cos(35^0)}$ and $y=\frac{\tan(75^0)}{\sin(25^0)}-\frac{1}{\cos(25^0)}$.

Which of the following is equivalent to $x\cdot y$?

A) $4\cos(40^0)\cos(75^0)$
B) $2\cos(40^0)\sec(75^0)$
C) $4\cos(20^0)\sec(75^0)$
D) $2\cos(40^0)\cos(25^0)$
E) $8\cos(20^0)\sec(75^0)$
F) None of these

14. Let Y be defined as $Y=\sum_{k=0}^{4}\cos(2kx)$ for any real number x .

Express the value of $\sum_{k=1}^{4}\cos^2(kx)$ in terms of m.

A) $Y+3$
B) $\frac{Y+3}{2}$
C) $Y+4$
D) $Y+2$
E) $\frac{Y+4}{2}$
F) None of these

15. Let x be a real number in $(0,\pi)$ satisfying $\dfrac{\sec(x)-1}{2\cot x}=\dfrac{-\tan^4 x}{\sec(x)+1}$. Evaluate:

$\sin(2x)+\sin^2\left(\dfrac{x}{2}\right)$.

A) $\dfrac{-3+2\sqrt{5}}{10}$

B) $\dfrac{-3-2\sqrt{5}}{10}$

C) $\dfrac{13+2\sqrt{5}}{10}$

D) $\dfrac{-2+3\sqrt{5}}{15}$

E) $\dfrac{-2-3\sqrt{5}}{15}$

F) None of these

16. Find the number of solutions of the equation

$\cos(5x)=\cos(3x)\cdot\cos(2x)$ over the interval $[0,2\pi]$.

A) 5
B) 8
C) 9
D) 11
E) 12
F) None of these

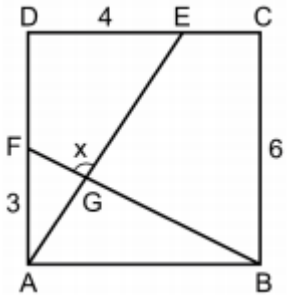
17. Let z be a complex number and $\bar{z}$ be its conjugate. Given: $4z-3\bar{z}=\dfrac{1-18i}{2-i}$,

write the expression $\dfrac{1}{z-2i}\cdot\left(z^2-8i^{19}\right)$ in the form of $a+bi$.

A) $12+9i$
B) $24-32i$
C) $\dfrac{108}{5}-\dfrac{19}{5}i$
D) $\dfrac{24}{5}-\dfrac{32}{5}i$
E) $\dfrac{12}{5}+\dfrac{9}{5}i$
F) None of these

18. A circle passing through the point $(0,10)$ is tangent to the $x-$axis at $x=20$. Find the radius of this circle.
    A) 18
    B) 20
    C) 21
    D) 25
    E) 30
    F) None of these

19. Let ABCD be a square as given on the figure below.

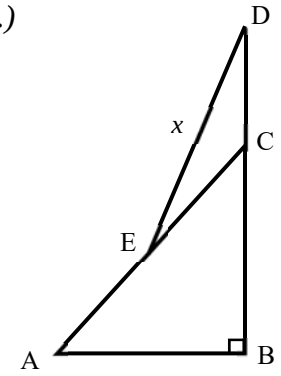


Given: $|DE|=4, |AF|=3, |BC|=6$ and $m(\angle FGE)=x$. Find the value of $\cot(x)$.

    A) -1/8
    B) -5/4
    C) -3/8
    D) -1/4
    E) -5/8
    F) None of these

20. Given: $\dfrac{\cot x}{\tan x+\cot x}=4\sin x-3$, find the value of $\sin x$.
    A) $3-2\sqrt{2}$
    B) $1-\sqrt{3}$
    C) $-1+\sqrt{2}$
    D) $-1+\sqrt{3}$
    E) $-2+2\sqrt{2}$
    F) None of these

21. Triangle ABC is an isosceles right triangle with right angle B. Given: E is the midpoint of the side AC and $|BD| = |AC| = 4$. Find $|DE| = x$. *(Note: The image is not drawn to scale.)*



A) $\sqrt{20-2\sqrt{2}}$

B) $2\sqrt{8-5\sqrt{2}}$

C) $2\sqrt{5-\sqrt{2}}$

D) $4\sqrt{4-\sqrt{2}}$

E) $2\sqrt{5-2\sqrt{2}}$

F) None of these

22. Evaluate the following: $\sin\left(2\arcsin\left(\frac{3}{5}\right)\right)+\sec\left(2\arccos\left(\frac{\sqrt{2}}{3}\right)\right)$.

A) $-\frac{21}{25}$

B) $-\frac{31}{25}$

C) $\frac{19}{225}$

D) $\frac{269}{225}$

E) $-\frac{11}{25}$

F) None of these

23. Amy and Bob are standing at the seashore 2 miles apart. The coastline is a straight line between them. Both can see the same ship in the water. The angle between the coastline and the line between the ship and Amy is 45 degrees; the angle between the coastline and the line between the ship and Bob is 75 degrees. What is the distance between the ship and Bob?

A) $\sqrt{6}$ miles
B) $\frac{\sqrt{6}}{2}$ miles
C) $\frac{2\sqrt{6}}{3}$ miles
D) $\frac{2\sqrt{3}}{3}$ miles
E) $\frac{\sqrt{3}}{2}$ miles
F) None of these

24. Let t be a real number satisfying $\cos t = \tan t$. Find the value of the expression $\frac{2}{\sin t} + \cos^4 t - \sin t$.

A) 2
B) 3
C) -2
D) 1
E) 0
F) None of these

25. A regular polygon with side length $r\sqrt{2-\sqrt{3}}$ is inscribed in a circle with radius r . How many sides does this polygon have?

A) 6
B) 8
C) 9
D) 12
E) 18
F) None of these

26. Let $\vec{u}$ and $\vec{v}$ be two different vectors in the coordinate plane whose sum is equal to the zero vector. Which of the following is/are true?

I. $\left\|2\vec{u}-\vec{v}\right\| = \left\|\vec{v}\right\|$.

II. If $\vec{u}+3\vec{v} = (2,-4)$, then $\vec{u} = (-1,2)$.

III. The angle between $\vec{u}$ and $\vec{v}$ is 180^0.

A) I only
B) III only
C) I and II
D) I and III
E) II and III
F) None of these

27. Evaluate: $\left(1-\sqrt{3}i\right)^9$

A) $256-256\sqrt{3}i$
B) -1024
C) 512
D) -512
E) $1024-1024\sqrt{3}i$
F) None of these

28. Let $\theta \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$ and $m = \left(\cos\theta\right)^{\cos\theta}$, $n = \left(\sin\theta\right)^{\cos\theta}$, $p = \left(\cos\theta\right)^{\sin\theta}$, $q = \left(\cos\theta\right)^{\sin(4\theta)}$.

Which of the following is/are true?

I. $m < n < p$

II. $p < m < n$

III. $q - m < 0$

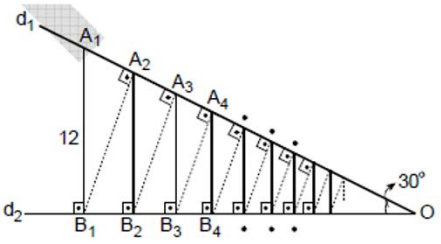
IV. $p < n < q$

A) II and III only
B) I and IV only
C) II and IV only
D) III only
E) II only
F) None of these

29. Drones *A* and *B* leave the same point at the same time. Drone *A* flies 10 miles with a bearing of $N75^0E$. Plane *B* flies 20 miles with a bearing of $S15^0W$. Find the distance between the drones *A* and *B*.

A) $10\sqrt{3}$
B) $10\sqrt{7}$
C) $20\sqrt{3}$
D) $16\sqrt{5}$
E) $20\sqrt{7}$
F) None of these

30. The lines d_1 and d_2 intersect at the point O with an angle of 30^0 as shown in the figure below.



Points $A_1, A_2, ..., A_{20}$ and $B_1, B_2, ..., B_{20}$ are marked with the following pattern:
A line segment perpendicular to d_2 is drawn from the point A_1 to the point B_1;
A line segment perpendicular to d_1 is drawn from the point B_1 to the point A_2;
A line segment perpendicular to d_2 is drawn from the point A_2 to the point B_2; and so on, until all 40 points are marked. *(The figure above is not complete due to limited space.)*
If $|A_1B_1| = 12$, find the sum: $|A_1B_1| + |A_2B_2| + |A_3B_3| + ... + |A_{20}B_{20}|$.

A) $\dfrac{3\left(2^{42} - 3^{21}\right)}{2^{40}}$
B) $\dfrac{3\left(2^{40} - 3^{20}\right)}{2^{38}}$
C) $\dfrac{3\left(1 - 3^{20}\right)}{2^{40}}$
D) $\dfrac{3\left(2^{40} - 3^{20}\right)}{2^{36}}$
E) $\dfrac{2^{42} - 3^{21}}{2^{40}}$
F) None of these

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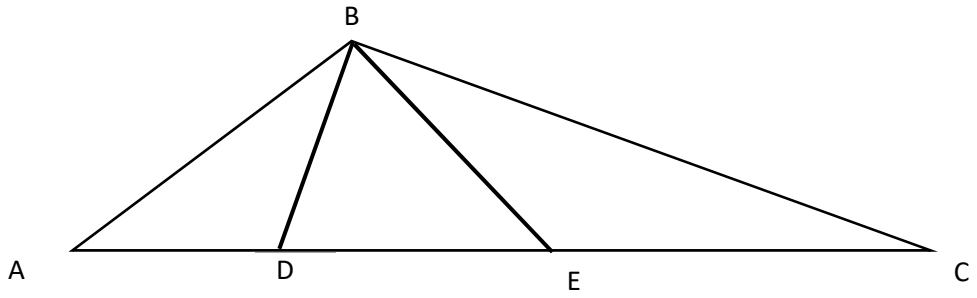
The following questions are part of this test, but they are not multiple choice. For the following 3 questions, write your answer on the answer sheet as a number. For example:

25, 0, 4.5, -2.7, $1+5\sqrt{7}$ **,** $4\sqrt{11}+5\sqrt{7}$ **,** $4\sqrt{3}$ **, 1/4 , 12/13 or 50/11**

are acceptable answers. Radical expressions should be reduced; for example, $\sqrt{12}$ **should be written as** $2\sqrt{3}$ **. Show your work on the empty space below each question and <u>write your final answer on the answer sheet</u>. Your work may be used to break ties.**

31. On the figure below, ABC and DBE are triangles with $|AD| = |BD|$ and $|BE| = |EC|$. Given $|AB| = 36\sqrt{2}$, $\sin(\angle BAD) = \frac{1}{3}$ and $\sin(\angle BCE) = \frac{1}{5}$, find the area of the triangle BDE.
(Note: The image is not drawn to scale.)



ANSWER: _____

32. The height of water in a bay various with time and can be modeled by the function $f(t) = a\sin(t) + b\cos(t)$, where a and b are real numbers and $t > 0$ represents time. The difference between the height of the waves at low tide and high tide is 10 feet. Find the largest possible value of $a + b$.

ANSWER: _____

33. Let $f(x) = \dfrac{x^5 + x^4 + 25x^2 + 20x + 10}{x^5 - 8x^3 + 2x^2 - 8x + 4}$.

If this function intersects its horizontal asymptote, then list the $x-$ coordinates of all points of intersection (if any). If it does not intersect the horizontal asymptote, state "none".

ANSWER: _____

THE END! Check that you wrote your answers on the answer sheet; only the answer sheet will be graded.