UNIVERSITY OF HOUSTON
HIGH SCHOOL MATHEMATICS CONTEST
Spring 2019 Calculus Test

NAME: ________________________________________________

SCHOOL: ______________________________________________

1. \( \lim_{x \to \infty} \frac{5 + 2x - x^3}{\sqrt{4x^5 + 2x^3 + x + 1}} = \)
   
   (a) \( \frac{1}{2} \)
   
   (b) 0
   
   (c) \(-1\)
   
   (d) \(-\frac{1}{2}\)
   
   (e) The limit does not exist.

2. \( \lim_{x \to 0} 3 \sin \left[ \frac{3 \sin(2\pi x)}{8x} \right] = \)
   
   (a) \(-3\sqrt{2}/2\)
   
   (b) \(-3\sqrt{3}/2\)
   
   (c) \(3\sqrt{2}/2\)
   
   (d) \(3/2\)
   
   (e) The limit does not exist.

3. \( \lim_{h \to 0} \frac{\cos(\pi/6 + 4h) - \cos(\pi/6)}{h} = \)
   
   (a) \(-2\)
   
   (b) \(2\sqrt{3}\)
   
   (c) 2
   
   (d) 0
   
   (e) \(-2\sqrt{3}\)
4. Set
\[ g(x) = \begin{cases} x^3 - 2x & x < 2, \\ ax^2 + bx & x \geq 2. \end{cases} \]
If \( g \) is everywhere differentiable, then \( a + b = \)
(a) 4
(b) −6
(c) 2
(d) 10
(e) −2

5. Three graphs labeled I, II and III are shown in the figure. One is the graph of \( f \), one is the graph of \( f' \) and one is the graph of \( f'' \). Which of the following correctly identifies each of the three graphs?

\[ f \quad f' \quad f'' \]
(a) I II III
(b) I III II
(c) II I III
(d) II III I
(e) III II I

6. Let \( f \) and \( g \) be differentiable functions which satisfy the following conditions:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>−1</td>
<td>−2</td>
<td>−1</td>
<td>0</td>
</tr>
</tbody>
</table>

If \( h(x) = f(g(x)) \), then \( h'(0) = \)
(a) −4
(b) −8
(c) 0
(d) 12
(e) 8
7. If \( f(x) = 4^x \ln(3e^x) \), then \( f'(0) = 
\begin{align*}
\text{(a)} & \quad 4 \ln(2) + 4 \\
\text{(b)} & \quad \ln(4) \ln(3) + 1 \\
\text{(c)} & \quad \ln(4) \ln(3) + 4 \\
\text{(d)} & \quad \ln(12) + 1 \\
\text{(e)} & \quad 4 \ln(7) + 1 
\end{align*}

8. When the linear approximation of \( f(x) = \sqrt{4 + \ln x} \) near \( x = 1 \) is used, an estimate of \( f(1.08) \) is:
\begin{align*}
\text{(a)} & \quad 1.98 \\
\text{(b)} & \quad 2.01 \\
\text{(c)} & \quad 2.02 \\
\text{(d)} & \quad 2.04 \\
\text{(e)} & \quad 2.06 
\end{align*}

9. Suppose that \( f \) is continuous on \([0, 4]\) and differentiable on \((0, 4)\). Suppose also that \( f(0) = 5 \) and \( f(4) = -3 \). Which of the following statements is not necessarily true?

I. There exists a number \( c \in (0, 4) \) such that \( f'(c) < -1 \)
II. There exists a number \( c \in (0, 4) \) such that \( f(c) = \pi \).
III. There exists a number \( c \in (0, 4) \) such that \( f'(c) = 2 \).
IV. If \( c \in (0, 4) \), and \( f'(c) = 0 \), then \( f(c) \) is either a maximum of a minimum of \( f \) on \([-1, 3]\)
\begin{align*}
\text{(a)} & \quad \text{II, IV} \\
\text{(b)} & \quad \text{I, II} \\
\text{(c)} & \quad \text{II, III, IV} \\
\text{(d)} & \quad \text{III, IV} \\
\text{(e)} & \quad \text{III only} 
\end{align*}

10. The position of a particle moving along a horizontal line is given by
\[
s(t) = \int_0^t (u^2 - 6u + 5) \, du, \quad 0 \leq t \leq 10,
\]
where \( t \) represents time. The interval(s) on which the speed of the particle is increasing is(are):
\begin{align*}
\text{(a)} & \quad (1, 3) \text{ and } (5, 10) \\
\text{(b)} & \quad [0, 1) \text{ and } (3, 5) \\
\text{(c)} & \quad (3, 10) \\
\text{(d)} & \quad [0, 1) \text{ and } (3, 10) \\
\text{(e)} & \quad (1, 5)
\end{align*}
11. A point \((x, y)\) is moving along a curve \(y = f(x)\). At the instant when the slope of the curve is \(-\frac{2}{3}\), the \(x\)-coordinate of the point is decreasing at the rate of 5 units per second. The rate of change, in units per second, of the \(y\)-coordinate is

(a) \(\frac{15}{2}\)
(b) \(-\frac{10}{3}\)
(c) \(\frac{3}{10}\)
(d) \(-\frac{2}{15}\)
(e) \(\frac{10}{3}\)

12. A function \(f\) is differentiable and decreasing on \((-\infty, \infty)\). If \(g(x) = f(x^3 - 3x^2 - 9x)\), then \(g\) has a local maximum at:

(a) \(x = 3\)
(b) \(x = 1\)
(c) \(x = 0\)
(d) \(x = -1\)
(e) There is no local maximum.

13. An equation for the normal line to the curve \(2x^3 + 2y^2 = 5xy\) at the point \((1, 2)\) is

(a) \(3x - 4y = 2\)
(b) \(3x + 4y = -5\)
(c) \(3x + 4y = 11\)
(d) \(4x - 3y = -2\)
(e) \(4x + 3y = 10\)

14. In square units, the area of the region bounded above by the the line \(y = 4\), below by the graph of \(f(x) = \frac{4}{1 + x^2}\), and on the sides by the lines \(x = \pm 1\) (see the figure), is:

(a) \(4 - \frac{\pi}{4}\)
(b) \(8 - \frac{\pi}{2}\)
(c) \(8 - \pi\)
(d) \(8 - 2\pi\)
(e) \(2\pi - 4\)
15. The base of a solid is the region in the first quadrant of the $xy$-plane bounded by $x^2 = 4y$, the $y$-axis and the line $y = 2$. Each plane section of the solid perpendicular to the $y$-axis is a semi-circle. The volume of the solid in cubic units is:

(a) $\frac{\pi}{2}$
(b) $2\pi$
(c) $\pi^2$
(d) $4\pi$
(e) $\pi$

16. The region bounded by the graph of $f(x) = \sqrt{x - 1}$, the vertical line $x = 10$, and the $x$-axis is revolved about the line $y = 3$. The volume of the generated solid, in cubic units, is:

(a) $\frac{99\pi}{2}$
(b) $\frac{189\pi}{2}$
(c) $\frac{135\pi}{2}$
(d) $\frac{119\pi}{2}$
(e) $\frac{137\pi}{2}$

17. If length is measured in centimeters, then the length of the graph of $f(x) = \ln(\sec x)$, where $0 \leq x \leq \pi/3$, is:

(a) $3 + \sqrt{2}$ cm
(b) $\ln \left(2 + \sqrt{3}\right)$ cm
(c) $\ln \left(\sqrt{3}\right)$ cm
(d) $2 + \sqrt{3}$ cm
(e) $\ln \left(\frac{1 + \sqrt{3}}{2}\right)$ cm

18. A curve in the plane has the property that the normal line to the curve at each point $P(x, y)$ always passes through the point $(0, 2)$. Find an equation for the curve given that it passes through the point $(3, 1)$.

(a) $x^2 + (y - 2)^2 = 10$
(b) $(x - 2)^2 + 2y^2 = 3$
(c) $y - 2 = x^2 - 10$
(d) $(x - 2)^2 + y^2 = 2$
(e) $\frac{x^2}{3} + 2(y - 2)^2 = 5$
19. Let $f$ be a continuous function on $(-\infty, \infty)$. If $F(x) = \int_2^x x^2 f(t) \, dt$, then $F''(2) =$

(a) $4f(2) + 4f'(2)$
(b) $8f(2) + 4f'(2)$
(c) $6f(2) + 4f'(2) + 2f''(2)$
(d) $8f'(2) + 4f''(2)$
(e) $4f(2) + 8f'(2)$

20. The function $F(x) = 2x + \int_x^4 \sqrt{4 + 3t} \, dt$ has an inverse. $(F^{-1})'(4) =$

(a) $1/6$
(b) $-1/12$
(c) $1/18$
(d) $-1/14$
(e) $-1/18$

21. Find $k$ if the average value of $f(x) = x^3 + 1$ on $[0, k]$ is 17.

(a) 6
(b) $\sqrt[4]{68}$
(c) 4
(d) 3
(e) $\sqrt{48}$

22. $\int_1^2 \frac{x^2}{\sqrt{4 - x^2}} \, dx$ is equivalent to:

(a) $4 \int_1^2 \sin^2 \theta \, d\theta$
(b) $2 \int_0^{\pi/2} \sin \theta \tan \theta \, d\theta$
(c) $2 \int_0^{\pi/2} \frac{\sin^2 \theta}{\cos \theta} \, d\theta$
(d) $4 \int_0^{\pi/2} \sin^2 \theta \, d\theta$
(e) $4 \int_{\pi/6}^{\pi/2} \sin^2 \theta \, d\theta$
23. If \( \int_0^8 e^x \, dx = A \), then \( \int_0^2 x^2 e^{x^3} \, dx = \)

(a) \( \frac{1}{3} A \)
(b) \( A^3 \)
(c) \( \frac{1}{3} A^3 \)
(d) \( 3A \)
(e) \( A \)

24. Find the number(s) \( a \) such that \( \lim_{x \to 0} \frac{e^{a^2 x^2} - \cos(4x)}{x^2} = 12. \)

(a) \( a = \pm \sqrt{12} \)
(b) \( a = 2, 4 \)
(c) \( a = \pm \sqrt{20} \)
(d) \( a = -4, 4 \)
(e) \( a = -2, 2 \)

25. The general solution of the differential equation \( \frac{dy}{dx} = 1 - 2x \frac{y}{y} \) is a family of:

(a) straight lines
(b) circles
(c) ellipses
(d) parabolas
(e) hyperbolas

26. The rate at which a certain bacteria population grows is proportional to number of bacteria present. Initially there were 1,000 bacteria present and the population doubled in 6 hours. Approximately how many hours will it take for the population to reach 10,000?

(a) 17
(b) 31
(c) 14
(d) 20
(e) 24

27. \( \lim_{x \to 0} (1 + 2x)^{1/x} = \)

(a) 0
(b) 2
(c) \( e \)
(d) \( e^2 \)
(e) The limit does not exist.
28. If a block of ice melts at the rate of \( \frac{72}{2t+3} \) cm\(^3\)/min, then the closest approximation to the amount of ice which melts during the first three minutes is:

(a) 40 cm\(^3\)
(b) 44 cm\(^3\)
(c) 36 cm\(^3\)
(d) 32 cm\(^3\)
(e) 48 cm\(^3\)

29. Set \( f(x) = \begin{cases} 2x + 1, & 1 \leq x < 3 \\ 4, & 3 \leq x \leq 5 \end{cases} \) If \( F(x) = \int_1^x f(t) \, dt \), then \( F(4) = \)

(a) 16
(b) 12
(c) 26
(d) 14
(e) 17

30. \( \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{n} \left( \frac{1}{1 + \frac{k}{n}} \right) = \)

(a) 2
(b) \ln 2
(c) 2\ln 2
(d) \ln 2 - 1
(e) 1