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University of Houston
High School Math Contest – 2019
Pre-Calculus Test

1. Let $f(x) = -2\ln(x)$, $g(x) = e^{2x+3}$, and $h(x) = -2^x$. If $a = (f \circ g \circ h)(2)$ and $b = h^{-1}\left(-\frac{1}{\sqrt{2}}\right) + \left(g^{-1} \circ f^{-1}\right)(-4)$, find $a+b$.
   A) -23
   B) 10
   C) 12
   D) -11
   E) 9
   F) None of the above

2. Consider the function: $f(x) = \dfrac{\sqrt{2-\log(x+5)}}{x^2+5x-6}$. Find the number of integers that are in the domain of this function.
   A) 98
   B) 99
   C) 100
   D) 102
   E) 104
   F) None of these

3. Given: $\ln(2) = 0.7$, $\ln(3) = 1.1$, $\ln(5) = 1.6$, $\ln(7) = 1.9$.
   Let $A = \ln(324000)$, $B = \ln(0.081)$, and $C = \ln(115.2)$. Find the value of: $100A + 80B + 50C$.
   A) 1310
   B) 1710
   C) 1780
   D) 1280
   E) 1510
   F) None of these

4. Order the following numbers from smallest to greatest:

$$a = \sin\left(\frac{7\pi}{6}\right) + \cos\left(\frac{11\pi}{6}\right)$$
$$b = \sin\left(\frac{11\pi}{3}\right) + \cos\left(\frac{26\pi}{3}\right)$$
$$c = \sin\left(\frac{43\pi}{2}\right) + \cos(27\pi)$$
$$d = \tan\left(\frac{43\pi}{4}\right) + \csc\left(\frac{65\pi}{6}\right)$$
$$e = \sin\left(\frac{11\pi}{4}\right) + \cos\left(\frac{44\pi}{3}\right)$$

A) $c < b < e < d < a$
B) $c < b < a < e < d$
C) $b < c < e < a < d$
D) $b < c < a < e < d$
E) $c < b < e < a < d$
F) None of these

5. Evaluate: $\arcsin\left(-\frac{1}{2}\right) + \arccos\left(-\frac{\sqrt{3}}{2}\right) + \arctan(-1) + \arctan\left(\sqrt{3}\right)$

A) $\frac{11\pi}{12}$
B) $\frac{2\pi}{3}$
C) $\frac{3\pi}{4}$
D) $\frac{\pi}{4}$
E) $\frac{7\pi}{12}$
F) None of these

6. If $\sec(x) = 1.2$ and $\csc(x) = 1.8$, evaluate the expression: $\tan^2(x) + \csc x \tan x + \sec x \cot x$.

A) 3.44
B) 4.44
C) 4.56
D) 3.64
E) 4.08
F) None of these.

7. Triangles ABC and DBC share the common base BC. In ΔDBC, $\angle DCB$ is an obtuse angle. The following information is given about the side lengths: $AB = 4,\ AC = 6,\ DC = 2\sqrt{14}$. If $m(\angle A) = 60^0$ and $m(\angle D) = 30^0$, find the area of ΔDBC.

A) $28\sqrt{3} + 4$
B) $14\sqrt{3}$
C) $6\sqrt{7}$
D) $7\left(1+\sqrt{3}\right)$
E) $6\left(1+\sqrt{3}\right)$
F) None of these

8. If $0 < t < \dfrac{\pi}{2}$ and $\cos(t) = \dfrac{4}{5}$; evaluate: $\dfrac{\sin(t+15\pi)+\sin(t-11\pi)}{\tan(2t+5\pi)}$

A) $\dfrac{7}{20}$

B) $\dfrac{3}{4}$

C) $-\dfrac{23}{25}$

D) $-\dfrac{9}{20}$

E) $-\dfrac{7}{20}$

F) None of these

9. Consider the functions: $f(x)=\dfrac{\cos(x)}{2\sin^2(x)-\sin(x)-1}$ and $g(x)=\dfrac{1+\sin(x)}{4\cos^2(x)-3}$.

Find the smallest positive real number that is not in the domain of either of these functions.

A) $\dfrac{\pi}{6}$
B) $\dfrac{5\pi}{6}$
C) $\dfrac{11\pi}{6}$
D) $\dfrac{7\pi}{6}$
E) $\dfrac{\pi}{2}$
F) None of these

10. Which of the following is equivalent to: $\left[\dfrac{\sin(-t)}{1+\cos(-t)}-\csc(-t)\right]\left[\tan(t)+\cot(t)\right]$

A) $1+\cot(t)$
B) $\csc(t)$
C) $\csc^2(t)$
D) $-\cot^2(t)$
E) $-\csc^2(t)$
F) None of these

11. Scientists believe that the average temperatures at various places on Earth vary from cooler to warmer over thousands of years. At one place on Earth, the highest average temperature is 80^0 and the lowest is 60^0. Suppose it takes 20,000 years to go from the high to low average and the average temperature was at a high point of 80^0 in year 2000. Set up a sinusoidal function $T(t)$ (where *t* is time in years) to model this phenomenon.

A) $T(t)=10\cos\left(\dfrac{\pi}{20{,}000}t\right)+70$
B) $T(t)=20\cos\left(\dfrac{\pi}{20{,}000}(t-2000)\right)+70$
C) $T(t)=20\cos\left(\dfrac{\pi}{10{,}000}(t-2000)\right)+70$

(The choices continue on the next page.)

D) $T(t) = 10\cos\left(\frac{\pi}{20{,}000}(t-2000)\right)+70$

E) $T(t) = 10\cos\left(\frac{\pi}{10{,}000}(t-2000)\right)+70$

F) None of these

12. Which of the following is equivalent to the expression given below?

$$\left[\frac{1}{\sec\theta+\tan\theta}+\frac{1}{\sec\theta-\tan\theta}\right]^2-\frac{8\csc(\theta)}{\sec(\theta)\sin(2\theta)}$$

A) $-4\tan^2(\theta)$

B) $-16\cot(2\theta)\csc(2\theta)$

C) $-4\tan(2\theta)\sec^2(2\theta)$

D) $-16\csc^2(2\theta)$

E) $-16\cot(2\theta)\csc^2(\theta)$

F) None of the above

13. Which of the following is an x-intercept for the function: $f(x)=\dfrac{2-8\cos^2\left(\frac{x}{2}+\frac{\pi}{6}\right)}{1+\sin(x)}$?

A) $\frac{29\pi}{6}$

B) $\frac{13\pi}{4}$

C) 4π

D) $\frac{5\pi}{6}$

E) $\frac{13\pi}{3}$

F) None of these

14. Let $K = \cos\left(2\arcsin\left(\frac{2}{3}\right)\right)$, $L = \sin\left(2\arccos\left(\frac{1}{3}\right)\right)$, $M = \cot\left(3\arctan\left(\frac{1}{3}\right)\right)$

Find the value of: $K \cdot L \cdot M$.

A) $\frac{16\sqrt{2}}{243}$
B) $\frac{4\sqrt{2}}{117}$
C) $\frac{8\sqrt{5}}{117}$
D) $\frac{8\sqrt{2}}{39}$
E) $\frac{16\sqrt{5}}{243}$
F) None of these

15. Let $y = \arcsin\left(\frac{x}{2}\right)$ where $0 < x \le 0.1$. Which of the following is an expression for $\sin(3y)$ in terms of x?

A) $\frac{x\sqrt{4-x^2}}{2-x^2}$
B) $\frac{5x-2x^3}{4}$
C) $\frac{x\sqrt{4-x^2}}{2}$
D) $\frac{2x-x^3}{4}$
E) $\frac{3x-x^3}{2}$
F) None of these

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16. Given $\pi < \theta < 2\pi$ with $\cos\theta = \dfrac{1}{4}$, find the value of $\tan\left(\dfrac{\theta}{2}\right) - \tan\left(2\theta\right)$.

A) $\dfrac{-12\sqrt{15}}{35}$

B) $\dfrac{-2\sqrt{15}}{35}$

C) $\dfrac{12\sqrt{15}}{35}$

D) $\dfrac{5\sqrt{15}}{14}$

E) $\dfrac{2\sqrt{15}}{35}$

F) None of the above

17. The following equation is solved over the interval $\left[0,\pi\right)$: $3\sin(4x)+5=6$.

If the smallest solution in that interval is $x=a$; find the largest solution in this interval.

A) $\pi - a$

B) $\dfrac{3\pi}{4}+a$

C) $\dfrac{\pi}{2}+a$

D) $\dfrac{\pi}{4}+a$

E) $\dfrac{3\pi}{4}-a$

F) None of these

18. Over the interval $[0,2\pi]$, how many x-intercepts does the following function have?

$$f(x)=2\cos^2(2x)-3\cos(2x)+1$$

A) 3
B) 4
C) 6
D) 7
E) 8
F) None of these

19. Two cars leave point A at the same time. Blue car heads due east of point A, and red car travels in a direction 60^0 south of east of point A. Blue car's speed is 40 mph, and red car's speed is 30 mph. Assume they travel in these directions for 2 hours. At the end of 2 hours, both make a U-turn immediately (assume no time is lost, no distance is covered during the U-turn) and each travel in the opposite direction with twice their original speed for 15 minutes, and then stop. Find the distance between these cars when they stop.

A) $15\sqrt{13}$
B) $20\sqrt{3}$
C) $10\sqrt{11}$
D) $10\sqrt{17}$
E) $15\sqrt{11}$
F) None of these

20. For $\frac{\pi}{2} < \theta < \pi$, which of the following is equivalent to the expression given below?

$$\frac{\tan(\theta)-\cot(\theta)}{\tan(\theta)+\cot(\theta)}+\frac{\cos(\theta)}{\sqrt{1+\tan^2(\theta)}}+\frac{\cot^2(\theta)}{(1-\csc(\theta))(1+\csc(\theta))}$$

A) $-3\cos^2(\theta)$
B) $-\cos^2(\theta)$
C) $-\sin^2(\theta)$
D) $-\cos(2\theta)+1$
E) $\sin^2(\theta)$
F) None of these

21. The function $f(x) = A\sin(2x)+B$ (where A and B are positive real numbers) has a minimum value of -2 and a maximum value of 10. Consider the function defined as $g(x) = 3f(x)+B\sin(x)\cos(x)+2A$. Find the maximum value of $g(x)$. $f'(x)$

A) 28
B) 44
C) 40
D) 22
E) 36
F) None of these

22. Given: $0 < y < x < \pi < z < 2\pi$; $\sin(x) = \frac{4}{5}$, $\cos(y) = -\frac{4}{5}$, $\tan(z) = \frac{12}{5}$. Find:

$\frac{\sin(x-y)}{\cos(y+z)} = ?$

A) $\frac{124}{325}$
B) $-\frac{91}{40}$
C) $-\frac{13}{40}$
D) $-\frac{23}{120}$
E) $\frac{14}{65}$
F) None of these

23. Given that the angle between the vectors $\vec{u} = \langle 1, k \rangle$ and $\vec{v} = \langle k^2 + 2,\ 3 \rangle$ is 90^0; find the smallest possible value of k.

A) 1
B) -1
C) 2
D) -2
E) 0
F) None of these

24. Consider the circles $x^2 + 4x + y^2 + 2y - 20 = 0$ and $x^2 - 20x + y^2 - 8y = -67$. Let r_1, r_2 be the radii of these circles and d be the distance between the centers of these circles. Find: $r_1 + r_2 + d$.

A) $12 + 4\sqrt{6}$
B) 30
C) 25
D) 22
E) $10 + 5\sqrt{2}$
F) None of these

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25. All points of intersections of the conic sections $x^2+y^2=4$ and $\frac{x^2}{16}+y^2=1$ are colored with red.

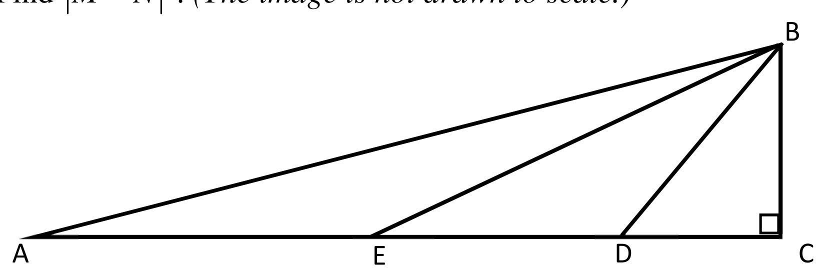
All points of intersections of the conic sections $x^2-x+y^2=0$ and $x-10y^2=0$ are colored with blue. Draw line segments that start at a blue point and end at a red point. How many such line segments can be drawn?

A) 0
B) 4
C) 8
D) 12
E) 15
F) None of these

26. Given the conic section: $25x^2+16y^2-50x+96y=231$; label the vertices as V_1 and V_2.
Given the conic section: $x^2+y^2-16x+2y=-64$; label the center as C. A triangle is drawn with vertices: C, V_1 and V_2. What is the area of this triangle?

A) 40
B) 35
C) 80
D) 70
E) 28
F) None of these

27. In the figure below, ΔABC is a right triangle. The following information is given about side lengths: $BC=4, AC=20, AE=BE, DE=BD$. Let M be the area of ΔBDE and N be the area of ΔBEA. Find $|M-N|$. *(The image is not drawn to scale.)*



(The choices are given on the next page.)

A) $\frac{149}{25}$
B) $\frac{169}{15}$
C) $\frac{143}{15}$
D) $\frac{191}{30}$
E) $\frac{104}{15}$
F) None of these

28. Let $\{s_n\}$ be a sequence defined as: $s_1 = \sin\left(\frac{41\pi}{6}\right)$; $s_n = 2\cos\left(\frac{5n\pi}{6}\right) + s_{n-1}$ for $n > 1$. Find the sum of the first 10 terms of this sequence.

A) $8\sqrt{3} - 5$
B) $8\sqrt{3} - 6$
C) $10\sqrt{3} - 6$
D) $12\sqrt{3} - 4$
E) $10\sqrt{3} - 5$
F) None of these

29. Consider the equation: $\sin(3x)\cos(x) = \sin(7x)\cos(5x)$. How many solutions does this equation have over the interval $[0, \pi)$?

A) 15
B) 7
C) 9
D) 12
E) 13
F) None of these

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The following questions are part of this test, but they are not multiple choice. For the following 4 questions, write your answer on the answer sheet as a number. For example:

25, 0, 4.5, -2.7, $1+5\sqrt{7}$ **,** $4\sqrt{11}+5\sqrt{7}$ **,** $4\sqrt{3}$ **, 1/4 , 12/13 or 50/11**

are acceptable answers. Radical expressions and fractions should be reduced. Show your work on the empty space below each question and write your final answer on the answer sheet. Your work may be used to break ties.

30. Given $\cos(4x)=\dfrac{1}{5}$, find the value of M that satisfies the equation:

$$4\left(\sin^8(x)-\cos^8(x)\right)=M\cos(2x)$$

(Continue to the next page.)

31. Let $f(x) = -4x^6 + 2x^5 + 6x^4 - 7x^3 + 2x^2 + 6x - 3$ and $g(x) = 4 - \sec^4(\pi x)$.

Let x_1 and x_2 be the smallest two positive x-intercepts of $f(x)$ $(x_1 < x_2)$;

Let x_3 and x_4 be the smallest two positive x-intercepts of $g(x)$ $(x_3 < x_4)$.

Find $10x_1 + 60{x_2}^2 + 20x_3 + 60x_4$.

32. The numbers $1, 2, 3, \ldots, 16$ are split into two groups $A = \{a_1, a_2, \ldots, a_8\}$ and $B = \{b_1, b_2, \ldots, b_8\}$ such that:

(i) The sums of all numbers in each group are equal; $\sum_{i=1}^{8} a_i = \sum_{i=1}^{8} b_i$.

(ii) The sums of squares of all numbers in each group are equal: $\sum_{i=1}^{8} {a_i}^2 = \sum_{i=1}^{8} {b_i}^2$.

(iii) The sums of cubes of all numbers in each group are equal: $\sum_{i=1}^{8} {a_i}^3 = \sum_{i=1}^{8} {b_i}^3$.

If the numbers in each set are ordered from least to greatest and if $a_1 < b_1$, find:

$a_3 + 3b_4 + {a_5}^2 + 2b_7$.

(Continue to the next page.)

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33. Let $x = \cot\left(\frac{\pi}{10}\right)$; $y = \cot\left(\frac{\pi}{8}\right)$; $z = \cot\left(\frac{11\pi}{40}\right)$.

Suppose: $x + y + z = 6.3$ and $(x + y)^2 - (x - y)^2 = 29.7$. Use this information to find the value of z. *(State your answer as a simple fraction.)*

THE END. Write your answers on the answer sheet; only the answer sheet will be graded.