University of Houston
High School Mathematics Contest
Spring 2020 Calculus Test

Name: ____

School: ____

1. $\lim\limits_{x\to\infty} \dfrac{5x^3e^{-x} + 4x^2 + 3x + 2}{\sqrt{1+x^4}} =$

   (a) 2
   (b) 3
   (c) 4
   (d) 5
   (e) This limit does not exist.

2. $\lim\limits_{h\to 0} \dfrac{\tan(\pi/4 + h) - \tan(\pi/4)}{3h} =$

   (a) 6
   (b) 2/3
   (c) 3
   (d) 1/3
   (e) This limit does not exist.

3. Suppose $f(x)$ and $g(x)$ are two differentiable functions on the interval $(0,2)$ that satisfy the two conditions

$$f(x) + g(x) = f(x) \cdot g(x) \tag{1}$$
$$f'(1) = -g'(1) \neq 0 \tag{2}$$

What is necessarily true about the numbers $f(1)$ and $g(1)$?

(a) $f(1) = g(1)$
(b) $f(1) = -g(1)$
(c) $f(1) \cdot g(1) = 1$
(d) $f(1) = 0$ and $g(1) = 0$
(e) $f(1) = 2$ and $g(1) = 2$

4. We are given that the following limit holds for *some* function $f(x)$:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(2 + \frac{i}{n}\right) \frac{1}{n} = 1.$$

Of the following options provided below, which, if any, could equal the function $f(x)$?

(a) $f(x) = e^x$
(b) $f(x) = \frac{\pi}{2}\cos(\pi x)$
(c) $f(x) = \frac{\pi}{2}\sin(\pi x)$
(d) $f(x) = 2x - 1$
(e) None of the above

5. For which values of a and b (amongst the options provided below) will $f(x) = ax^3 + bx^2 + (a+b)x$ have a relative minimum at $x = 0$?

(a) $a = 0, b = -\pi$
(b) $a = \pi, b = -\pi$
(c) $a = \pi, b = \pi$
(d) $a = -\pi, b = \pi$
(e) $a = -\pi, b = -\pi$

6. Suppose $f(x)$ is a continuous function that satisfies

$$4x^2 - \frac{a}{2}x^4 \leq f(x) \leq \frac{a^2+4}{a}x^2$$

where $a \geq 0$ is a constant. For what value of a is it true that

$$\lim_{x \to 0} \frac{f(x)}{\sin^2 x} = 4?$$

(a) $a = 4$
(b) $a = 3$
(c) $a = 2$
(d) $a = 1$
(e) There is no value of a that makes this equation true.

7. A particle is moving along the x-axis with velocity

$$v(t) = \frac{t}{\sqrt{1+t^2}} \text{ m/s.}$$

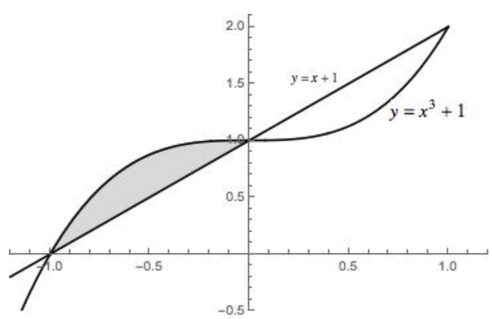
How far will the particle have traveled between $t = 1$ second and $t = 7$ seconds?

(a) $2\sqrt{2}$ meters
(b) $1/\sqrt{2}$ meters
(c) 1 meter
(d) $4\sqrt{2}$ meters
(e) None of the above.

8. Two functions $j(x)$ and $k(x)$ pass through the same point $(2, 5)$ but are approximated by different tangent lines. The function $j(x)$ is approximated by $3x - 1$ and $k(x)$ is approximated by $x + 3$. The equation for the line tangent to $g(x) = j(x)/k(x)$ at $x = 2$ is

   (a) $y - 2 = 0$
   (b) $3y - x - 1 = 0$
   (c) $5y - 2x - 1 = 0$
   (d) $2y + x - 4 = 0$
   (e) None of the above.

9. The region between the two graphs $y = x + 1$ and $y = x^3 + 1$ and lying in Quadrant II is shown in the image below.



   Which of the following expressions represent the area of this region?

   I. $\displaystyle\int_{-1}^{0} x^3 - x\,dx$

   II. $\displaystyle\int_{0}^{1} y^{1/3} - y\,dy$

   III. $\displaystyle\int_{-1}^{0} \frac{x^3 - x}{2}\,dx + \int_{0}^{1} \frac{x - x^3}{2}\,dx$

   (a) I only
   (b) I, II only
   (c) I, III only
   (d) II, III only
   (e) I, II, III

10. Evaluate the limit

$$\lim_{b \to 0} \int_0^b \frac{\sqrt{1+\sin(x^2)}}{\sin b}\, dx =$$

(a) 1

(b) 0

(c) $\sqrt{2}$

(d) e

(e) This limit does not exist.

11. Suppose we are told that $\lim_{x \to 0} f(x) \cdot \sin\left(\frac{1}{x}\right)$ exists. Which, if any, of the following statements can be true?

I. $f(x) = x$

II. $f(x) = \cot\left(\frac{1}{x}\right)$

III. $f(x) = \csc\left(\frac{1}{x}\right)$

(a) I only.

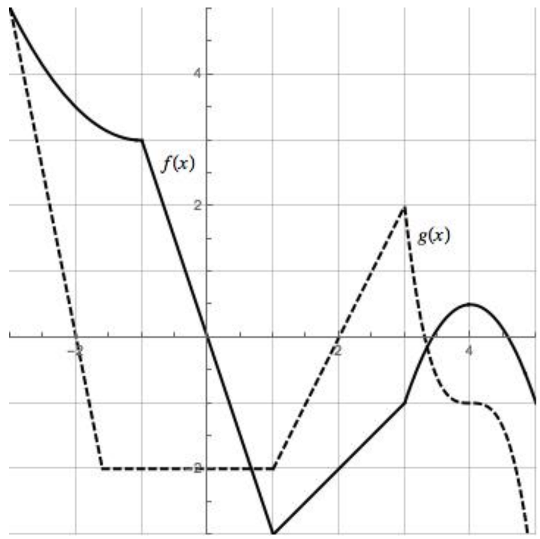
(b) II only.

(c) III only.

(d) I and II only.

(e) I and III only.

12. Graphs of the piecewise differentiable functions $f(x)$ and $g(x)$ are shown below.



Given $h(x) = f(g(x))$, the value of $h'(2)$ is

(a) -5
(b) -6
(c) 0
(d) 5
(e) undefined

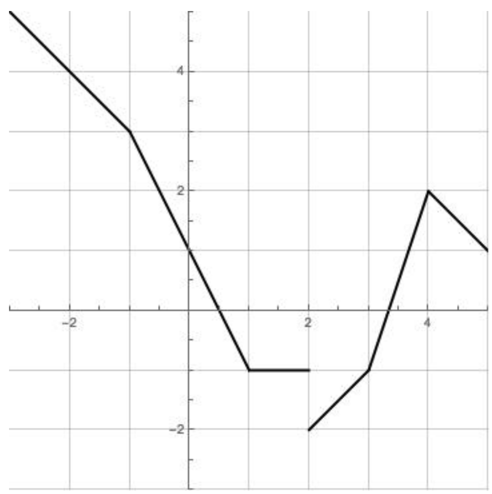
13. The equation for the normal line to the curve $x^3 - y^3 + xy^2 + x^{-1} = 2$ at the point $(1, 1)$ is given by

(a) $y + 3x - 3 = 0$
(b) $3y - x - 1 = 0$
(c) $y - 3x + 2 = 0$
(d) $3y + x - 4 = 0$
(e) None of the above.

14. The length of the portion of the graph $f(t) = \sqrt{4 - t^2}$ that lies over the interval $0 \leq t \leq 2$ equals

(a) π
(b) $\dfrac{\pi}{2}$
(c) 4π
(d) 0
(e) None of the above

15. A graph of the function $y = f(x)$ consists entirely of line segments and is shown below.



Based on this graph, $\displaystyle\int_{-3}^{5} |\, f(x)\,| - f(x)\, dx =$

(a) 35/6
(b) 149/6
(c) 19/2
(d) 46/3
(e) None of the above

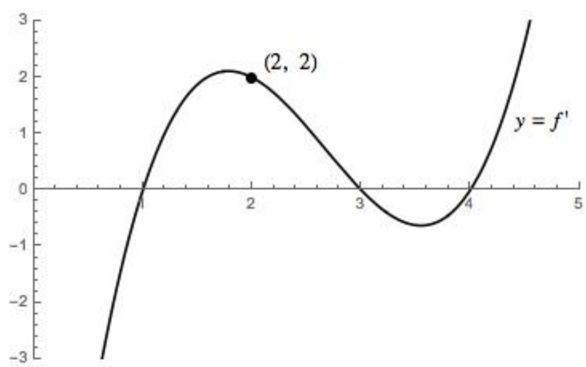
16. Consider the function

$$f(x) = \frac{(1+2x)^{\frac{1}{2}}(1+4x)^{\frac{1}{4}}\cdots(1+2nx)^{\frac{1}{2n}}}{(1+3x)^{\frac{1}{3}}(1+5x)^{\frac{1}{5}}\cdots(1+(2m+1)x)^{\frac{1}{2m+1}}}$$

where n and m are positive integers satisfying $n+m-20 = f'(0) = 2010$. Then $n =$

(a) 2020
(b) 2019
(c) 1920
(d) 1
(e) 0

17. The graph of the first derivative f' of a function f is shown below. Which of the following statements must be true?



I. $f(1) > f(3)$
II. $f(3) > f(4)$
III. $f(3) - f(2) \leq 2$
IV. $f(3) - f(2) \geq 2$

(a) II only
(b) I, IV only
(c) II, III only
(d) I, II, III only
(e) I, II, IV only

18. The graph of $\ln x$ passes through the point (x_0, y_0) where its tangent line is parallel to the secant line passing through $(1,0)$ and $(2, \ln 2)$. The value of $x_0 =$

(a) $\ln 2$

(b) $1/\ln 2$

(c) $\ln(1/2)$

(d) $-\ln(\ln 2)$

(e) e

19. The invertible function $F(x)$ is defined by

$$F(x) = \int_1^{e^x} \sqrt{3+t^4}\, dt.$$

The slope of the line tangent to the graph of F^{-1} at the point $(0,0)$ equals

(a) $1/2$

(b) $1/4$

(c) 1

(d) 2

(e) None of the above

20. If $f'(x) = e^{-x}(x+2)(x-5)^2$, then which of the following statements is necessarily true?

(a) f has a local minimum at $x=-2$ and f has a local maximum at $x=5$.

(b) f has an inflection point at $x=-2$ and f has a local maximum at $x=5$.

(c) f has a local minimum at $x=-2$ and f has an inflection point at $x=5$.

(d) f has a local maximum at $x=-2$ and f has a local minimum at $x=5$.

(e) f has an inflection point at $x=-2$ and f has a local minimum at $x=5$.

21. Which point or points on the graph of $y = 4 - x^2$ is/are closest to the point $(0, 2)$?

   (a) $(0, 4)$
   (b) $\left(-\frac{\sqrt{6}}{2}, \frac{5}{2}\right), \left(\frac{\sqrt{6}}{2}, \frac{5}{2}\right)$
   (c) $(-2, 0), (2, 0)$
   (d) $(-\sqrt{2}, 2), (\sqrt{2}, 2)$
   (e) $\left(-\frac{\sqrt{3}}{2}, \frac{13}{4}\right), \left(\frac{\sqrt{3}}{2}, \frac{13}{4}\right)$

22. Suppose f is a function that is differentiable at every point in the real line, is strictly decreasing on its domain. Which, if any, of the following statements about the function $g(x) = f\left(x^4 + x^2 + 1\right)$ are necessarily true?

   (a) $g(x)$ is increasing on $(-\infty, \infty)$
   (b) $g(x)$ has a relative maximum at $x = 0$ and a relative minimum at $x = -1$
   (c) $g(x)$ has a relative minimum at $x = 0$ and a relative maximum at $x = -1$.
   (d) $g(x)$ has a relative minimum at $x = -1$.
   (e) None of the above.

23. Evaluate the definite integral

$$\int_0^{\pi/2} \frac{(\sin\theta)^{2020}}{(\sin\theta)^{2020} + (\cos\theta)^{2020}}\, d\theta =$$

   (a) 2020π
   (b) $\pi/2$
   (c) 1010π
   (d) $\pi/4$
   (e) None of the above

24. Consider the function $f(x) = a|x|\sin x + b\,|\cos(x/2)|$ where a and b are constants. Among the options provided below, which values of a and b ensure that the average value of $f(x)$ over $[-\pi, \pi]$ equals $\frac{2}{\pi}$?

(a) $a = 4, b = 0$

(b) $a = -1, b = -1$

(c) $a = 2020, b = 1$

(d) $a = 1, b = 2020$

(e) $a = \frac{1}{\pi}, b = \frac{1}{2}$

25. Suppose f is a differentiable function on $[0, 2]$ that satisfies $f(0) = f(2)$. Which of the following statements must be true?

I. $f(c) = 0$ for some c in $(0, 2)$.
II. $f'(c) = 0$ for some c in $(0, 2)$.
III. $f(c) = f(c - 1)$ for some c in $[1, 2]$

(a) I, II only

(b) I, III only

(c) II, III only

(d) II only

(e) I, II, III

26. $\lim\limits_{x\to\infty} \left(\dfrac{x+1}{x}\right)^{x/2} =$

(a) 1

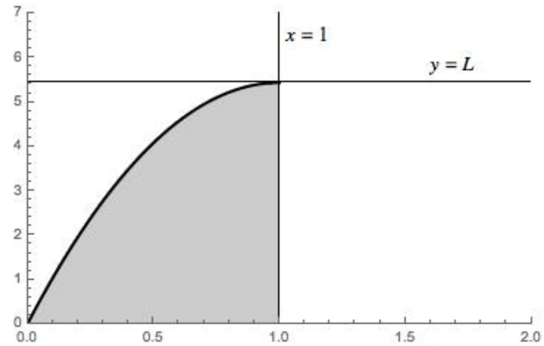
(b) e

(c) $\sqrt{e}$

(d) e^2

(e) This limit does not exist.

27. The region bounded by the parabola $y = L \cdot x \cdot (2 - x)$ and by the lines $y = 0$ and $x = 1$ is shown below.



When this region is revolved about the line $y = 0$ a solid of revolution is obtained, and when this region is revolved about the line $x = 1$ another solid of revolution is obtained. For which value(s) of $L \geq 0$ do these two solids have the same volume?

(a) $L = 0$
(b) $L = 0, L = 15/16$
(c) $L = 0, L = 15/14$
(d) $L = 0, L = 13/12$
(e) None of the above.

28. Given that $\int_0^1 xf(x)\,dx = A$ and that $\int_0^1 x^{2\pi-1} f(x^\pi)\,dx = 1$, it follows that $\cos A =$

(a) -1
(b) 0
(c) $\sqrt{2}/2$
(d) 1
(e) None of the above

29. If $F(x)$ is an antiderivative for $f(x) = \sqrt{2x+1}$ and $F(0) = 1$, then $F(3/2) =$

(a) 3

(b) 10/3

(c) 8/3

(d) π

(e) 2

30. For this question you can set a equal to the numerical value of the correct answer choice from question 29. With such a choice in mind evaluate the definite integral

$$\int_a^{a^2} \frac{dx}{x \ln x} =$$

(a) $\ln 2$

(b) $\ln 3$

(c) $\ln \pi$

(d) $\ln(10/3)$

(e) $\ln(8/3)$