NAME_____ SCHOOL_____

University of Houston

High School Math Contest – 2020

Pre-Calculus Test

1. Given: $g(x)=5e^{4x-1}$ and $h(x)=\log_2(4x+3)$, evaluate $g^{-1}(5)+h^{-1}(3)$.

A) 5/4
B) 5
C) $5e$
D) 3/2
E) 1
F) None of the above

2. Given:

$$a=\sin\left(\frac{31\pi}{6}\right)-\cos\left(\frac{49\pi}{3}\right),$$
$$b=\sin\left(\frac{201\pi}{4}\right)+\cos\left(\frac{1683\pi}{4}\right),$$
$$c=\sin\left(-\frac{91\pi}{2}\right)-\cos(-43\pi),$$
$$d=\tan\left(\frac{65\pi}{4}\right)+\csc\left(\frac{65\pi}{6}\right).$$

Find the value of: $(b^3-1)(c^3-d^3)(a^3-1)$.

A) 0
B) 38
C) -7
D) 7
E) -38
F) None of the above

3. Let $m = \arcsin\left(-\frac{1}{2}\right)$, $n = \arccos\left(-\frac{1}{2}\right)$, $p = \arctan\left(-\sqrt{3}\right)$. Find the value of: $\frac{m}{2} + \frac{n}{4} + \frac{p}{6}$.

A) $\frac{\pi}{36}$

B) $\frac{\pi}{18}$

C) $\frac{\pi}{12}$

D) $\frac{5\pi}{36}$

E) $\frac{5\pi}{12}$

F) None of the above

4. Let $f(x) = \ln\left(8x^3\right)$ and $g(x)$ be the inverse of this function. Let $a = g\left(g^2\left(0\right)\right)$; find the value of $\ln\left(2a\right)$.

A) $\frac{1}{2}$

B) $\frac{1}{4}$

C) $\frac{1}{12}$

D) $\frac{1}{6}$

E) $\frac{3}{4}$

F) None of the above.

5. Find the average of all integers that are in the domain of the following function:

$$f(x) = \frac{\ln\left(64 - x^2\right)}{\sqrt{x^3 - 2x^2 - 8x}}$$

A) 6

B) 4.25

C) 5.75

D) 3.375

E) 5.25

F) None of the above

6. Let *D* be the point on the conic section $25x^2+50x+4y^2-16y=59$ that is farthest away from the *x*-axis; let *E* be the point on the conic section $4x^2-16x+9y^2-18y=11$ that is farthest away from the *y*-axis. Find the distance between the points *D* and *E*.

A) $2\sqrt{5}$
B) $6\sqrt{2}$
C) $\sqrt{71}$
D) $5\sqrt{2}$
E) $\sqrt{82}$
F) None of the above

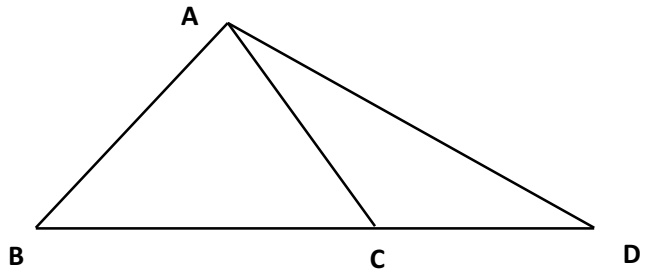
7. Consider the graphs of: $x=y^2-2y$ and $x^2+(y-1)^2=1$. At how many points do these graphs intersect (if any)?

A) 0
B) 1
C) 2
D) 3
E) 4
F) None of the above

8. Given that the angle between the vectors $\vec{u}=\langle 1,2k\rangle$ and $\vec{v}=\langle k^2-8,\ 1\rangle$ is 90^0; find the largest possible value for the magnitude of $\vec{u}$.

A) $3\sqrt{2}$
B) $2\sqrt{5}$
C) $\sqrt{17}$
D) $\sqrt{33}$
E) $\sqrt{65}$
F) None of the above

9. Given: ΔABC and ΔACD are triangles with: $m(\angle ACB) = 60^0$, $AC = 8$, $BC = 10$, $\tan(\angle ADC) = \frac{\sqrt{3}}{4}$. Find the area of ΔABD.



A) $22\sqrt{3}$
B) 46
C) $44\sqrt{3}$
D) $36\sqrt{3}$
E) 22
F) None of the above

10. ΔABC is a triangle with $m(\angle A) = 60^0$, $m(\angle B) = 45^0$. If $AC = 20$, find the length of the side BC.

A) $10\sqrt{2}$
B) $20\sqrt{2}$
C) $20\sqrt{3}$
D) $\frac{20\sqrt{6}}{3}$
E) $10\sqrt{6}$
F) None of the above

11. Find the largest solution for the following equation over the interval $\left[0, \frac{\pi}{2}\right)$:

$$3\sin(4x) + 5 = 6$$

A) $x = \pi - \frac{1}{4}\arcsin\left(\frac{1}{3}\right)$

B) $x = \pi - \frac{1}{4}\arcsin\left(\frac{2}{3}\right)$

C) $x = \frac{\pi}{4} - \arcsin\left(\frac{1}{3}\right)$

D) $x = \pi - \arcsin\left(\frac{1}{12}\right)$

E) $x = \frac{\pi}{4} - \frac{1}{4}\arcsin\left(\frac{1}{3}\right)$

F) None of the above

12. Find the value of: $\sin\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) + \arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$.

A) $\frac{\sqrt{6}+\sqrt{2}}{4}$

B) $\frac{\sqrt{2}-\sqrt{6}}{4}$

C) $\frac{\sqrt{6}-\sqrt{2}}{4}$

D) $\frac{-\sqrt{6}-\sqrt{2}}{4}$

E) $\frac{\sqrt{6}+2\sqrt{2}}{4}$

F) None of the above

13. Let a and b be the largest two solutions for the following equation over the interval $[0,3\pi]$: $2\cos^2(2x)+3\sin(2x)=3$. Find the value of $a+b$.

A) $\frac{14\pi}{3}$

B) $\frac{29\pi}{12}$

C) $\frac{23\pi}{6}$

D) $\frac{16\pi}{3}$

E) $\frac{29\pi}{3}$

F) None of the above

14. Given θ is an acute angle with $\tan(\theta)=10$, find the value of $\frac{\sin(2\theta)}{\cos(3\theta)}$.

A) $-\frac{29\sqrt{101}}{299}$

B) $-\frac{40\sqrt{101}}{119}$

C) $-\frac{20\sqrt{101}}{101}$

D) $-\frac{20\sqrt{101}}{119}$

E) $-\frac{20\sqrt{101}}{299}$

F) None of the above

15. Which of the following is NOT in the domain of $f(x)=\dfrac{1-\sin(x)}{4-2\sec(4x)}$?

A) $x=\dfrac{7\pi}{6}$

B) $x=\dfrac{5\pi}{6}$

C) $x=\dfrac{2\pi}{3}$

D) $x=\dfrac{23\pi}{12}$

E) $x=\dfrac{\pi}{2}$

F) None of the above

16. Given θ is an acute angle with $\cos\theta=\dfrac{1}{4}$, find the value of $\tan\left(\dfrac{\theta}{2}\right)+\tan\left(2\theta\right)$.

A) $\dfrac{2\sqrt{15}}{35}$

B) $\dfrac{12\sqrt{15}}{35}$

C) $\dfrac{5\sqrt{15}}{14}$

D) $\dfrac{-4\sqrt{15}}{35}$

E) $\dfrac{\sqrt{15}}{35}$

F) None of the above

17. If $0 < t < \frac{\pi}{2}$ and $\cos(t) = \frac{4}{5}$; evaluate: $\sin(2t + 20\pi) + \sin(t - 5\pi) + \cot\left(\frac{t}{2} + 5\pi\right)$

A) $\frac{84}{25}$

B) $\frac{52}{75}$

C) $\frac{42}{75}$

D) $\frac{111}{100}$

E) $\frac{114}{25}$

F) None of the above

18. Consider the following equation over the interval $[0, 2\pi)$: $4\sin^2(3x) + 2 = 3$.

Find the sum of the largest three solutions in this interval.

A) $\frac{11\pi}{3}$

B) $\frac{25\pi}{6}$

C) $\frac{17\pi}{3}$

D) $\frac{95\pi}{18}$

E) $\frac{59\pi}{18}$

F) None of the above

19. Let $f(x) = A\sin(2x) + D$ and $g(x) = M\sin(4x) - 2N$ where A and D are positive, M and N are negative real numbers. If the maximum values of $f(x)$ and $g(x)$ are both 16 and the minimum values are both 4, find the value of $f\left(\frac{\pi}{12}\right) + g\left(\frac{N\pi}{24}\right)$.

A) 0

B) 13

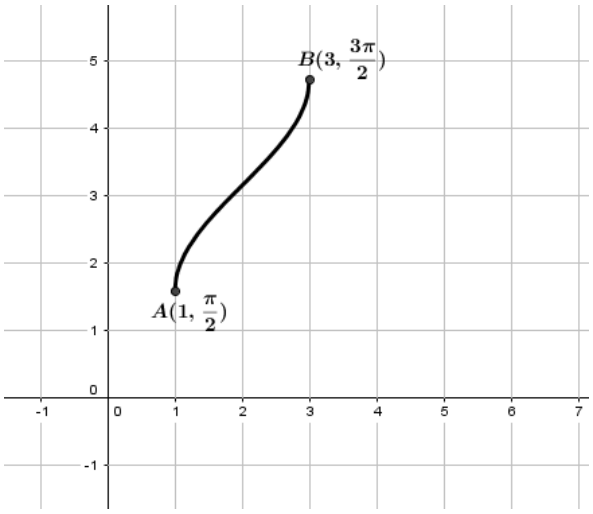
C) 26

D) 22

E) 44

F) None of the above

20. The graph of a function is given below. Which of the following can be this function?



A) $f(x) = \arcsin(x-2) + \frac{\pi}{2}$

B) $f(x) = \arcsin(x-1) + 2$

C) $f(x) = \arccos(x-2) + 1$

D) $f(x) = 2\arcsin(x-2)$

E) $f(x) = \arcsin(x-2) + \pi$

F) None of the above

21. Let $a = \cos^2(x) + \cos^2\left(\frac{2\pi}{3} + x\right) + \cos^2\left(\frac{2\pi}{3} - x\right)$ and

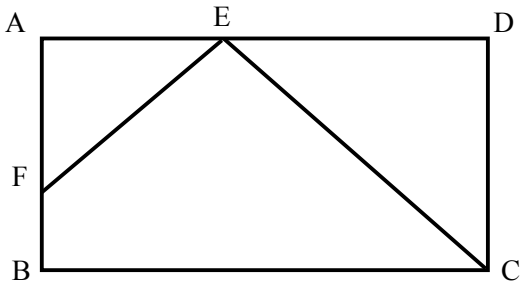
$b = \sin^2\left(\frac{\pi}{3} + x\right) + \sin^2\left(\frac{\pi}{3} - x\right) - \cos^2(x)$. Find the value of $5a - 3b$

A) 5
B) 3
C) 6
D) 8
E) 1
F) None of the above

22. Evaluate: $\sin\left(2\arccos\left(\frac{3}{5}\right)\right) - \cos\left(2\arcsin\left(\frac{1}{5}\right)\right)$

A) $\frac{4}{25}$
B) $\frac{1}{25}$
C) $\frac{27}{25}$
D) $\frac{7}{5}$
E) $\frac{3}{25}$
F) None of the above.

23. In the figure below, $ABCD$ is a rectangle with: $8BF = 4AF = BC$, $m(\angle FEC) = 90^0$, $m(\angle BFE) = \theta$. Find the sum of all possible values of $\cot(\theta)$. *(The image is not drawn to scale.)*



A) $-\frac{11}{3}$
B)
C) $-\frac{8}{3}$
D) $-\frac{20}{3}$
E) $-\frac{13}{6}$
F) None of the above

24. Let $a = \sqrt[3]{6\sin(5x) + 6\cos(4y) + 8\cos(5x) + 14}$, where x, y are real numbers. Find the average of all possible **integer** values of a.

A) 2
B) 1.25
C) 2.25
D) 1
E) 2.5
F) None of the above

25. Given: $\arctan(x-1) + \arctan(2x+1) = \frac{\pi}{4}$, where x is a real number. Find the largest possible value for $4x^2 + 4x + 1$.

A) 5
B) 25
C) 10
D) 6
E) 36
F) None of the above

26. Given: $0 < x < \frac{\pi}{8}$ and $\tan(4x) = \frac{12}{5}$, find the value of the following expression:

$$\cot(x) + \csc(6x)$$

A) $\frac{23\sqrt{13}+18}{46}$

B) $\frac{13\sqrt{13}+69}{36}$

C) $\frac{36\sqrt{13}+69}{46}$

D) $\frac{18\sqrt{13}+23}{46}$

E) $\frac{18\sqrt{13}+39}{46}$

F) None of the above

27. Assume that $\tan(10^0) = a$ and $\sec(10^0) = b$. Which of the following would be equivalent to the expression: $\frac{\cos(40^0)+\cos(50^0)-1}{\cos(40^0)-\cos(50^0)+1}$?

A) $\frac{a}{1-b^2}$

B) $\frac{-2a}{b^2-2}$

C) $\frac{b^2}{2a}$

D) $\frac{2a+b^2}{b^2}$

E) $\frac{-2a+b^2}{b^2-2}$

F) None of the above

28. Let a be the largest and b be the smallest $x-$ intercepts of the function $f(x)=x^7-2x^6-53x^5+10x^4+244x^3-8x^2-192x$. Find the value of: $4a-2b$.

A) 32
B) 42
C) 48
D) 36
E) 44
F) None of the above

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The following questions are part of this test, but they are not multiple choice. For the following 4 questions, write your answer on the answer sheet. If the answer is a number, the following are examples of acceptable ways of writing your answer:

25, 0, 4.5, -2.7, $1+5\sqrt{7}$, $4\sqrt{11}+5\sqrt{7}$, $4\sqrt{3}$, $\frac{5\pi+2}{7}$, $-\frac{5\pi}{7}$, $\frac{13+5\sqrt{7}}{4}$, 1/4, 12/13, 50/11

Radical expressions and fractions should be reduced. Show your work on the empty space below each question and write your final answer on the answer sheet. Your work may be used to break ties.

29. Consider the solutions for the following equation over the interval $[-\pi, 0]$:

$$\cos\left(2x-\frac{\pi}{3}\right)=\sin\left(x-\frac{\pi}{4}\right).$$

Find the sum of the smallest two solutions in this interval. *Write your answer as a fraction.*

30. Find the value of the following expression:

$$\frac{\cos^2(20^0)+\cos^2(50^0)-2\cos(20^0)\cos(50^0)\cos(70^0)}{\cos^2(200^0)+\sin^2\left(250^0\right)}$$

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31. Assume that $\tan\left(2^0\right)=0.03$. Use this information to find the value of the following expression: $\dfrac{\tan\left(1^0\right)\tan\left(61^0\right)}{\tan\left(3^0\right)\tan\left(31^0\right)}+\dfrac{\tan\left(61^0\right)+\tan\left(31^0\right)}{1-\tan\left(61^0\right)\tan\left(31^0\right)}$

Write your answer as a fraction.

32. Let ΔABC be a triangle with $m\left(\angle A\right)=120^0$. The following information is given about the lengths of the sides of this triangle: $BC=x^2+x+1$ and $AC=2x+1$, where x is a positive real number. Express the length of the side AB in terms of x.

THE END. Write your answers on the answer sheet; only the answer sheet will be graded.