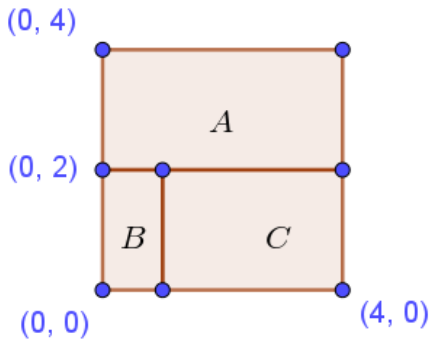
Algebra II Exam

University of Houston Math Contest 2024

1. $c > 0$ and the roots of $x^2 + 27x + c = 0$ are numbers a and b. Which of the following must be true.

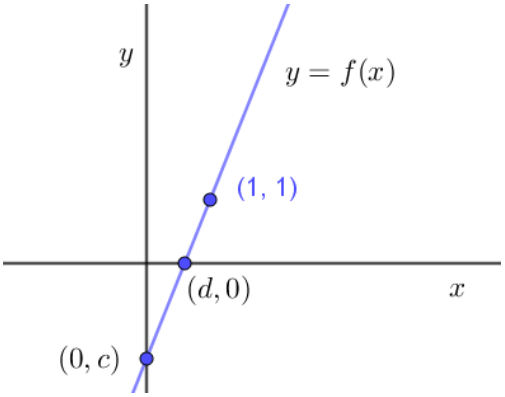
   a. $a + b = 27$
   b. a and b are positive numbers
   c. $c + ab = 0$
   d. $27 - 4c > 0$
   e. $a - b + c = 0$
   f. None of these.

2. Three rectangles are shown below with areas A, B and C, and $BC = 12$. The rectangles B and C might not be drawn to scale. Find $|B - C|$.



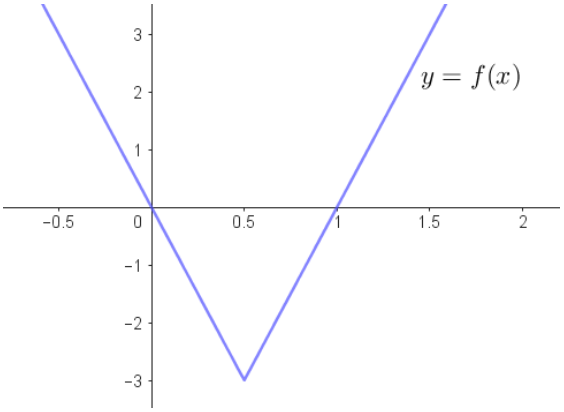
   a. 4.5
   b. 5
   c. 6
   d. 4
   e. 5.5
   f. None of these.

3. $f(x)$ is an invertible function whose graph is the line shown in the picture below. In addition, the slope of the line is a and $|ad| = \frac{3}{2}$. Find $f^{-1}(2)$.



a. 1.2
b. 1.3
c. 1.4
d. 1.5
e. 1.6
f. None of these.

4. The graph of the function $f(x)$ is shown below.



Give the y-coordinate of the point of intersection in the second quadrant of $y = f(x)$ and the line through $(5, -4)$ and $(-3,4)$.

a. 1.2
b. 1.1
c. 1.3
d. 1.4
e. 1.5
f. None of these.

5. Let $f(x) = \left(\sqrt{x}+2\right)^2\left(\sqrt{x}-2\right)^2 + 6 - 2x$. Evaluate $f(2)$.

   a. $3\sqrt{2}+2$
   b. $5\sqrt{2}+2$
   c. 6
   d. $6\sqrt{2}+2$
   e. 4
   f. None of these.

6. A certain chemical is the product of a reaction. The volume of chemical that has been created by time $t \geq 0$ minutes after the reaction starts is given by

$$A(t) = 3 - 2e^{-0.1t} \text{ cm}^3.$$

   Give the time at which there will be twice as much of the chemical as there was at the start of the reaction.

   a. $10\ln(2)$ minutes
   b. $2\ln(10)$ minutes
   c. $8\ln(2)$ minutes
   d. $8\ln(10)$ minutes
   e. $5\ln(2)$ minutes
   f. None of these.

7. a, b and c are real numbers, and the x-intercepts of $f(x) = x^3 + ax^2 + bx + c$ are at $x = 2$, $x = -3$ and $x = 5$. Give $f(0)$.

   a. 32
   b. 28
   c. 26
   d. 24
   e. 30
   f. None of these.

8. The polynomial $p(x) = 3x^3 + ax^2 + bx + c$ has roots -3, $2+2i$ and $2-2i$. Give the value of $2a + 5b + c$.

   a. 5
   b. 6
   c. -4
   d. -3
   e. 7
   f. None of these.

9. Solve $2^{\log_2\left(\log_5\left(\sqrt[3]{2m+1}\right)\right)} = -1$ for m. Then give the y-intercept of the line with slope m that passes through the point $(1, -1)$.

    a. $-63/125$
    b. $-87/125$
    c. $-82/125$
    d. $-65/125$
    e. $-69/185$
    f. None of these.

10. Define

$$f(x) = \frac{3x^3 - 6x^2 - 3x + 6}{x^2 + 2x - 3}.$$

    Let a be the sum of the x-coordinates of the x-intercepts of $f(x)$, let b be the sum of the x-coordinates of the vertical asymptotes of $f(x)$, and let c be the sum of the x-coordinates where $f(x)$ is undefined and does not have a vertical asymptote. Give the value of $a + 3b + 2c$.

    a. -5
    b. -6
    c. -7
    d. -8
    e. -9
    f. None of these.

11. If a liters of a mixture that is 11% acid is mixed with b liters of a mixture which is 16% acid and 2 liters of a mixture that is 12% acid, it produces a solution that is 13% acid. However, when $1.9a$ liters of the mixture that is 11% acid is mixed with b liters of the mixture which is 16% acid and 2 liters of the mixture that is 12% acid, it produces a solution that is 12% acid. Give the value of a.

    a. $44/3$ liters
    b. 13 liters
    c. $41/3$ liters
    d. $40/3$ liters
    e. $43/3$ liters
    f. None of these.

12. Suppose a and b are real numbers. Find the value of $a + b$ for which the graphs of

$$f(x) = (a + 2b + 1)x + 2a - 2b + 1$$

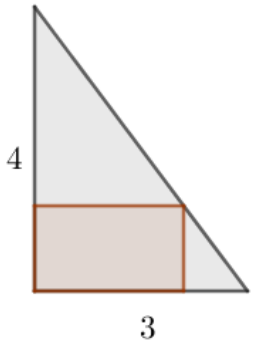
and

$$g(x) = (-3a + b - 2)x + a - b$$

are the same.

a. $2/5$
b. $-3/5$
c. $1/5$
d. $-4/5$
e. 1
f. None of these.

13. A rectangle is inscribed inside a right triangle with height 4 and base 3 as shown below.



Give the perimeter of this rectangle if it has the largest possible area.

a. 9
b. 6
c. 8
d. 7
e. 10
f. None of these.

14. Find the inverse of the function $f(x) = \frac{1}{2}x - 3$.

a. $g(x) = -2x + 3$
b. $g(x) = 2x + 3$
c. $g(x) = -2x + 6$
d. $g(x) = 2x + 6$
e. $f(x)$ is not invertible.
f. None of these.

15. Determine the smallest value of the function $f(x) = x^4 - 3x^2 - 4$.

a. $-13/2$
b. -6
c. $-25/4$
d. $-23/4$
e. $-11/2$
f. None of these.

16. The variable A varies inversely as the square of B, and B varies directly as the cube of C. If A equals 32 when C equals 2, what is the value of A when C equals 4?

a. 2
b. 4
c. $1/2$
d. $1/4$
e. There is not enough information.
f. None of these.

17. Rewrite the function $g(x) = 3x^2 - 4x + 5$ in the form $g(x) = a(x - h)^2 + k$, and give the value of $a + h + k$.

a. $20/3$
b. $19/3$
c. $17/3$
d. 5
e. $22/3$
f. None of these.

18. Give the value of a for which the following system of equations does not have a solution.

$$\begin{pmatrix} x - 2y + z = 3 \\ 2x - ay + 3z = 2 \\ -2x + 3y + z = 7 \end{pmatrix}$$

a. $17/3$
b. 6
c. $14/3$
d. 5
e. $16/3$
f. None of these.

19. a and b are real numbers and the focus of the parabola given by $y = 3x^2 + ax + b$ is at the point $(2,3)$. Give the value of b.

    a. $89/6$
    b. $179/12$
    c. 15
    d. $181/12$
    e. $91/6$
    f. None of these.

20. A circle begins enlarging, starting at $t = 0$, and it continues enlarging until $t = 1$. When $t = 0$, the radius is $r > 0$ (cm) and the area is A. The radius increases 6cm and the area increases $B\text{cm}^2$ by the time $t = 1$. Give the smallest value $b > 0$ so that $B > b$.

    a. 36π
    b. 30π
    c. 33π
    d. 39π
    e. 27π
    f. None of these.

21. Let a be the largest number so that the equation

$$\ln\left(\sqrt{a - x^2}\right) = \ln(6) - \ln(x)$$

    has exactly one solution. Give the value of $2a - 3$.

    a. 20
    b. 19
    c. 18
    d. 21
    e. 22
    f. None of these.

22. Give the largest number a so that the domain of $f(x) = \ln\left(3(2^x + 1) - \frac{1}{A}\right)$ is $(-\infty, \infty)$ when $A < a$.

    a. $1/21$
    b. $1/3$
    c. $1/9$
    d. 1
    e. 9
    f. None of these.

23. Which of the following functions are invertible?

I. $f(x) = 2x^3 + x^4$
II. $g(x) = \ln(x - 2)$
III. $h(x) = 2^{1-3x} + 6$
IV. $r(x) = x^2 + 3x + 1$
V. $s(x) = 3x^3 + x + 1$
VI. $F(x) = \sqrt[3]{3 - x} + 2$

a. II, III, V, VI
b. II, V, VI
c. I, II, III, V
d. II, V, VI
e. I, III, V, VI
f. None of these.

24. Determine the largest negative value of c so that the graphs of $f(x) = 2 - 3x$ and $g(x) = C|x - 1| - C$ intersect at more than one point when $c < C$.

a. -3.1
b. -3.2
c. -3
d. -2.9
e. -2.8
f. None of these.

25. The domain of the function $g(x) = \sqrt{6 - x - x^2}$ is an interval of the form $[a, b]$. Give the value below closest to b/a.

a. -0.42
b. -0.57
c. -0.69
d. -0.65
e. -0.60
f. None of these.

26. Let $f(x) = 3 - 2x$ and $g(x) = f^{-1}(x + 1)$. Find a and b so that the graph of $y = x^2 + ax + b$ passes through the points $(g(3),3)$ and $(2, g(2))$. Then give a/b.

a. $-24/14$
b. $27/12$
c. $-27/12$
d. $24/13$
e. $-27/13$
f. None of these.

27. The graphs of $f(x) = 3x + 7$ and $g(x) = x^2 + x + 1$ intersect at two values of x of the form $a + \sqrt{b}$ and $a - \sqrt{b}$. Give a/b.

    a. $2/7$
    b. $3/5$
    c. $1/3$
    d. $1/7$
    e. $3/7$
    f. None of these.

28. Determine the values of a and b for which the system

$$\begin{pmatrix} 2x + \frac{a}{y} = 3 \\ -3x + \frac{b}{y} = 2 \end{pmatrix}$$

    either does not have a solution or has more than one solution, and then give $\frac{b}{a} + \frac{a}{b}$.

    a. $-5/6$
    b. $-1/3$
    c. $-2/3$
    d. $-7/6$
    e. $-4/3$
    f. None of these.

29. Determine the domain $[a, b]$ of the function $f(x) = \sqrt{20 - 8x - x^2}$. Then give the slope of the line passing through the points $(3a, b)$ and $(3b, a)$.

    a. $-1/12$
    b. $-1/3$
    c. $-1/6$
    d. $-1/4$
    e. $-1/8$
    f. None of these.

30. Find the point that is a distance 4 from the point $(-12,10)$ and is closest to the line $x - 3y = 11$. Give the value below that is closest to the sum of the coordinates of this point.

    a. -4.9
    b. -4.4
    c. -4.8
    d. -4.7
    e. -4.5
    f. -4.3

31. Give the smallest value of $3x + y$, given that $x + 6y \geq 3$, $x \leq 4$ and $x, y \geq 0$.

    a. $1/2$
    b. $1/3$
    c. $1/4$
    d. $1/5$
    e. $1/6$
    f. None of these.

32. Let $f(x) = 3|x + 1| + 2$ and $g(x) = x^2 - 1$. There are distinct points (a, b) and (c, d) of intersection of the graphs of $H(x) = f\big(g(x)\big)$ and $f(x)$ with $a < c$. Give $f(c + a)$.

    a. 7
    b. 8
    c. 6
    d. 4
    e. 5
    f. None of these.

33. a is a number and $f(x) = x^3 + (4 - a)x^2 + (3 - 4a)x - 3a$. The graph of $f(x)$ passes through the points $(-3,0)$, $(-1,0)$ and $(1, -16)$. Give the value of a.

    a. $7/2$
    b. $4/3$
    c. $6/5$
    d. $5/3$
    e. $7/4$
    f. None of these.

34. Let $S = \{(x, y) \mid x \cdot y > 0\}$. The graph of $|y| = x^2 + 1$ does not represent a function, but the portion of the graph that lies in S represents a function. Call this function $f(x)$. The function $g(x) = f(x + 1) + f(x - 1)$ passes through the points $(a, 2)$ and $(b, 12)$. Find b/a.

    a. $7/4$
    b. 2
    c. 4
    d. $8/3$
    e. 3
    f. None of these.

35. Solve the system

$$\begin{pmatrix} \sqrt{2a+b-1} - \sqrt{2a+b-10} = 1 \\ a+b=10 \end{pmatrix}.$$

Give the value of a/b.

a. $-10/3$
b. $-7/3$
c. $-5/3$
d. $-4/3$
e. $-8/3$
f. None of these.

36. A triangle has one vertex at the point $(-4,2)$, one vertex on the line $y = 2x - 1$ and another vertex on the line $y = 2x + 1$. The vertices on the two lines have the same x-coordinate a. Let $g(a)$ be the area of the triangle. Solve the equation $g(a) = 2$ and give the sum of the solutions.

a. -4
b. -1
c. -3
d. -2
e. -5
f. None of these.