Calculus Exam - University of Houston Math Contest 2024

1. Evaluate the following limit:

$$\lim_{x\to\infty} x\sin\left(\frac{2024}{x}\right)$$

   a. 2024
   b. $1/2024$
   c. 0
   d. $\sin(2024)$
   e. This limit does not exist.
   f. None of the other answers

2. Consider the two parabolas $y = x^2$ and $y = -x^2 + 4x + b$ (where b is a constant whose value we forgot to tell you). If the two parabolas are tangent, then what is the value of b?
   a. $b = 1$
   b. $b = 0$
   c. $b = -2$
   d. $b = -1$
   e. There is no value of b for which the two parabolas are tangent.
   f. None of the other answers

3. A region, R, in the xy-plane is bounded by the curves $y = \cos^2(x)$, $y = -\sin^2(x)$, $x = 0$, and $x = b$ (where b is a positive constant whose value we forgot to tell you). If the region R has area equal to π square-units, then what is the value of b?
   a. $b = 2\pi$
   b. $b = 1$
   c. $b = \pi/2$
   d. $b = \sqrt{2}/2$
   e. There is no value of b for which the conditions are met.
   f. None of the other answers

4. Suppose that at every input x the differentiable function $f(x)$ satisfies

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{h},$$

   and has a graph that passes through the point $(2024,4)$. Evaluate the definite integral

$$\int_0^{506} f(x)dx.$$

   a. 506
   b. 2024
   c. 253
   d. $253/2$
   e. 0
   f. None of the other answers

5. Consider the function $f(x) = \begin{cases} e^x & \text{if} \quad x \geq 0 \\ \cos(x) & \text{if} \quad x < 0 \end{cases}$

Which, if any, of the following statements accurately describes this function?

a. $f(x)$ is continuous and differentiable at $x = 0$.
b. $f(x)$ is continuous at $x = 0$ but it is not differentiable at $x = 0$.
c. $f(x)$ is not continuous at $x = 0$ but it is differentiable at $x = 0$.
d. $f(x)$ is neither continuous nor differentiable at $x = 0$.
e. $f(x)$ is continuous at $x = 0$ and $f'(0) = 1$.
f. None of the other answers

6. Suppose $g(x)$ is an invertible, differentiable function whose tangent line at $x = 1$ is given by the equation $4x - 2y + 2 = 0$. Which, if any, of the following statements are true?
a. The graph of $g^{-1}(x)$ has a tangent line at $x = 3$ whose slope is $1/2$.
b. The graph of $g^{-1}(x)$ has a tangent line at $x = 3$ whose slope is $1/4$.
c. The graph of $g^{-1}(x)$ has a tangent line at $x = 1$ whose slope is $1/2$.
d. The graph of $g^{-1}(x)$ has a tangent line at $x = 1$ whose slope is $1/4$.
e. The graph of $g^{-1}(x)$ has a tangent line at $x = -1/3$ whose slope is $-1/2$.
f. None of the other answers

7. The function $f(x) = bx^2$ (where b is a positive constant whose value we forgot to tell you) has a graph along the interval $[0,1]$. When this graph is rotated about the x-axis, the resulting solid has a volume equal to 2π cubic-units. Determine the value of $f(1)$.
a. $f(1) = \sqrt{10}$
b. $f(1) = \sqrt{6}$
c. $f(1) = \sqrt{6\pi}$
d. $f(1) = \sqrt{\pi}$
e. The value of $f(1)$ cannot be determined from the information given.
f. None of the other answers

8. The graph of the continuous function, $y = f(x)$, lies in the region bounded by the lines $y = 5x + 3$ and $y = 6x - 2$. Evaluate $\lim_{x \to 5} f(x)$.
a. 28
b. 5
c. 2024
d. 2/5
e. The value of this limit cannot be determined from the given information.
f. None of the other answers

9. The point (a, b) lies *somewhere* in the xy-plane, but we forgot to tell you the values of both a and b, so you don't know *exactly* where it is. However, you *do* know that of all the points on the line $y + x = 1$ it is closest to $\left(\frac{1}{2}, \frac{1}{2}\right)$, and of all the points on the line $\mathrm{y} - \mathrm{x} = 2$ it is closest to $(2,4)$. Determine the values of a and b.

   a. $(a, b) = (3,3)$

   b. $(a, b) = (2,0)$

   c. $(a, b) = (2,2)$

   d. $(a, b) = (0,0)$

   e. $(a, b) = (0,2)$

   f. None of the other answers

10. Evaluate

$$\lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \frac{i}{n + i^2/n}$$

   a. $\ln(2)$

   b. $\pi/2$

   c. $\ln(\sqrt{2})$

   d. 0

   e. $\pi/4$

   f. None of the other answers

11. Evaluate the following definite integral:

$$\int_0^{\frac{1}{2}\ln(10)} \frac{e^{2x}}{\sqrt{15 + e^{2x}}}\, d\mathrm{x}$$

   a. 10

   b. 5

   c. 1/2

   d. 1

   e. 0

   f. None of the other answers

12. The function $g(t)$ has as its domain the entire real line, the function is increasing on the intervals $(-\infty,-1),(-1,0),(1,2)$ and $(2,2024)$, and it is decreasing on the intervals $(0,1)$ and $(2024,\infty)$. Which, if any, of the following statements accurately describe the critical points of g?

    a. There are local maxima when $t=-1,\ t=0,\ t\ =\ 2$, and there is a local minimum when $t=1$.

    b. There are local maxima when $t\ =\ -1$ and $t=0$, and there is a local minimum when $t=1$.

    c. There is a local maximum when $t\ =\ 0$, and there are local minima when $t=1$ and $t=2$.

    d. There are no local maxima, and there are local minima when $t=1,\ t=2$ and $t=2024$.

    e. There are no local maxima and there are no local minima.

    f. None of the other answers

13. Three positive numbers, x,y,z, are chosen from the real line in such a way that their sum equals 8 *and* so that the distance from z to $-x$ is 2. What is the maximum possible value of their product?

    a. 48

    b. $32\sqrt{3}/9$

    c. 18

    d. 6

    e. 3

    f. None of the other answers

14. Suppose the function $f(x)$ satisfies $|f(x)|\leq x^2$ for every real number x. Which, if any, of the following statements accurately describes $f(x)$?

    a. We can conclude that $f(x)$ is both continuous and differentiable at $x=0$ and that $f(0)=0$, but we cannot conclude anything about the value $f'(0)$.

    b. We can conclude that $f(x)$ is continuous at $x=0$, but we know nothing about the value $f(0)$, and $f(x)$ may or may not be differentiable at $x=0$.

    c. We can conclude that $f(x)$ is both continuous and differentiable at $x=0$ and that $f(0)=f'(0)=0$.

    d. We can conclude that $f(x)$ is continuous at $x=0$ with $f(0)=0$, but it may or may not be differentiable at $x=0$.

    e. We can conclude that $f(x)$ is both continuous and differentiable at $x=0$ and that $f(0)=1$, but we cannot conclude anything about the value $f'(0)$.

    f. None of the other answers

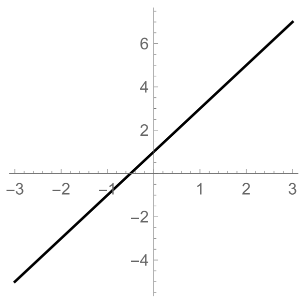
15. Suppose we are told the following limit holds:

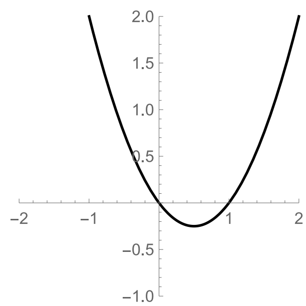
$$\lim_{n\to\infty} n\sqrt[n]{b} - n = 2024$$

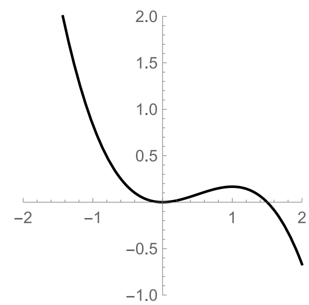
Determine the value of b.

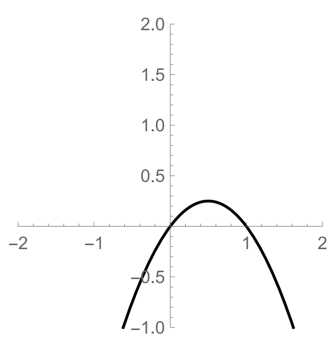
a. $b = 2024$
b. $b = \ln(2024)$
c. $b = e^{2024}$
d. $b = \log_{2024}(2)$
e. $b = \sqrt{2024}$
f. None of the other answers

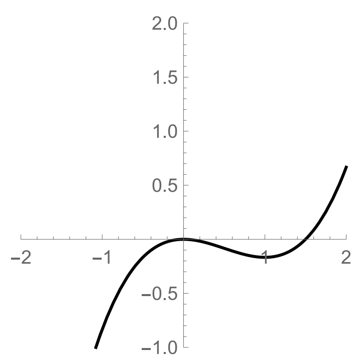
16. The graph of a function $f(x)$ is concave up on the interval $(0,1)$ and is concave down on the intervals $(-\infty, 0)$ and $(1, \infty)$. Among the options provided below, which could be the graph of $y = f'(x)$?

a. 

b. 

c. 

d. 

e. 

f. None of the other answers

17. The function $f(x) = \sin(x^3) + x^2$ along the interval $[-b, b]$ has an average value of 48. Again, we forgot to tell you the value of the positive number b, but that's okay since you have enough information to determine it yourself.

a. $b = \pi$
b. $b = 12$
c. $b = 2\sqrt[3]{4}$
d. $b = 3\sqrt{3}$
e. $b = 24$
f. None of the other answers

18. A particle is traveling along the x-axis in a rather peculiar way: its velocity depends on its position along the x-axis. More specifically, at time t seconds it is located at position x where it has velocity $\cos^2(x)$, and we know that if the particle were to travel *forever* it would eventually reach a position of $x = \pi/2$. At what point along the x-axis did the particle begin its journey?

   a. The particle began at $x = -\pi$.
   b. The particle began at $x = 0$.
   c. The particle began at $x = \pi/4$.
   d. The particle began at $x = -\pi/2$.
   e. The particle's initial position cannot be determined from the given information.
   f. None of the other answers

19. The graph of $y = e^x$ has a tangent line that passes through the origin when x equals which value?
   a. $x = 1$
   b. $x = e$
   c. $x = 0$
   d. $x = 2$
   e. $x = 1/e$
   f. None of the other answers

20. Suppose $f(x)$ is an invertible, differentiable function with *normal* line $y = m_1x + b_1$ at point (a, b), and suppose that the inverse function $f^{-1}(x)$ has a graph with tangent line $y = m_2x + b_2$ at the point (b, a). Which, if any, of the following statements must be true?

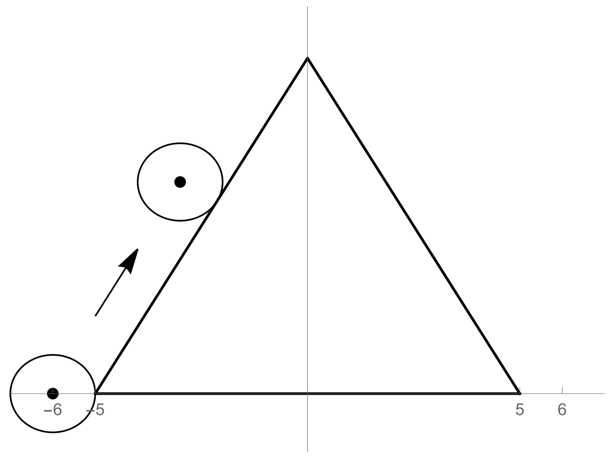
   a. $m_1m_2 = 1$
   b. $m_1 = m_2$
   c. $m_1m_2 = -1$
   d. $m_1 = -m_2$
   e. $m_1 + m_2 = 1$
   f. None of the other answers

21. The famous "Sine integral" is a function, $Si(x)$, defined as follows:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} \mathrm{dt}$$

   Evaluate $\lim_{x \to 0} \frac{\mathrm{Si}(x)}{\sin(x)}$.
   a. 0
   b. $\pi/2$
   c. 1
   d. -1
   e. The limit does not exist.
   f. None of the other answers

22. An equilateral triangle is arranged in the plane so that its vertices are located at $(-5,0)$, $(0,5\sqrt{3})$, and $(5,0)$, and a circle of radius 1 rolls along two legs of the triangle, beginning its journey at $(-5,0)$ and ending at $(5,0)$ (see the image below).

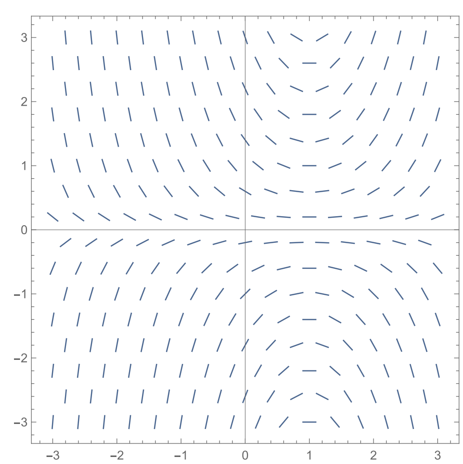


If we let $f(x)$ denote the height of the circle's center above point $x \in [-6,6]$, then what is the value of $f'(-2) + f'(0) + f'(4)$?

a. 0
b. 1
c. 5
d. $\sqrt{3}$
e. No value can be assigned to this sum since $f'(0)$ does not exist.
f. None of the other answers

23. Which of the following differential equations has the slope field shown below?

a. $\frac{dy}{dx} = y$
b. $\frac{dy}{dx} = x - 1$
c. $\frac{dy}{dx} = yx - y$
d. $\frac{dy}{dx} = yx + y$
e. $\frac{dy}{dx} = y^2 - y$
f. None of the other answers



24. A cylindrical piece of ice is melting in such a way that both its radius, r, and its height, h, decrease at a constant rate of b feet per second (where b is a positive real number whose value we forgot to tell you – oops). We also know that when both the radius and height equal $r = h = 1$ ft, the volume is decreasing at a rate of 6π cubic-feet per second. Determine the value of b.
    a. $b = 8$
    b. $b = 3$
    c. $b = 2$
    d. $b = 1$
    e. $b = 0$
    f. None of the other answers

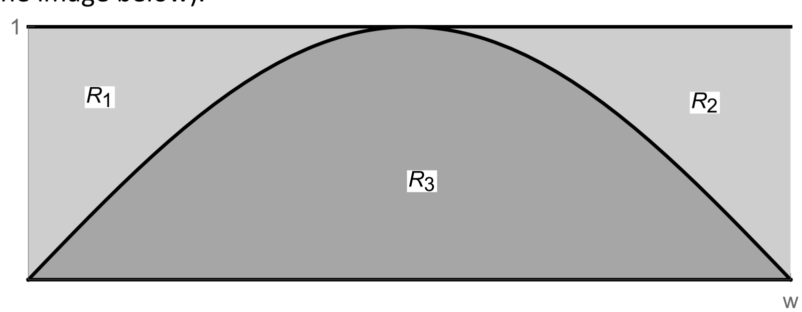
25. Consider the function $f\colon [-1,\infty) \to (-\infty,\infty)$ defined by

$$\mathrm{f}(x) = \int_{-1}^{x^2} \frac{2^t - 16}{1 + \ln(t+2)}\,\mathrm{dt}$$

Which, if any, of the following statements is true?

    a. $f(x)$ has a local maximum at $x = 2$ and no other critical numbers in its domain.
    b. $f(x)$ has a local minimum at $x = 2$ and no other critical numbers in its domain.
    c. $f(x)$ has a local minimum at $x = 0$, a local maximum at $x = 2$, and no other critical numbers in its domain.
    d. $f(x)$ has a local maximum at $x = 0$, a local minimum at $x = 2$, and no other critical numbers in its domain.
    e. $f(x)$ has no local maxima nor local minima.
    f. None of the other answers

26. Given a rectangle of width w and height 1 consider the three regions, R_1, R_2 and R_3, one obtains by using the sinusoidal curve joining the lower-left and lower-right corners of the rectangle (as shown in the image below).



Define the function $f(w)$ as follows: $f(w) = \dfrac{\text{Area of Region } R_3}{\text{Area of the Rectangle}}$

Finally, let n denote the number of questions you have thus far answered correctly on this exam. Evaluate $f(n)$.

    a. $2/\pi$ b. $3/\pi$ c. 0 d. 1
    e. This limit does not exist since the ratio approaches positive infinity.
    f. None of the other answers

27. The graph of a function $f(x)$ has a tangent line at $x = 2$ that passes through the points $(1,2)$ and $(3,6)$. Determine the value of $f'(2) + f(2)$.

   a. 4
   b. 8
   c. 6
   d. 3
   e. 0
   f. None of the other answers

28. Consider the piecewise function $f(x)$ defined as

$$f(x) = \begin{cases} ax + b & \text{if} \quad x < 1 \\ x^2 + cx + 1 & \text{if} \quad x \geq 1 \end{cases}$$

where $a, b,$ and c are constants. If we are told that Rolle's Theorem applies to $f(x)$ over the interval $[0,2]$, determine the value of $a + b + c$.

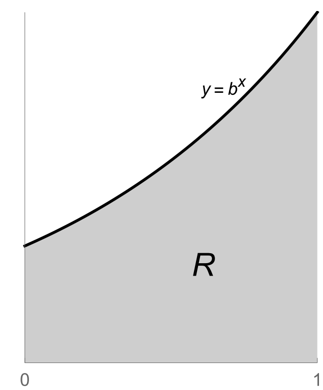
   a. 0
   b. -3
   c. -1/2
   d. 5
   e. 1
   f. None of the other answers

29. Determine the value of the positive integer n given that

$$\lim_{x \to 0} \left(\frac{1}{x}\right) \ln\left(\frac{(e^x + e^{2x} + e^{3x} + \cdots + e^{nx})(e^x + e^{2x} + e^{3x} + \cdots + e^{nx})}{n^2}\right) = 2024$$

   a. $n = 2025$
   b. $n = 2024$
   c. $n = 2023$
   d. $n = 2022$
   e. $n = 2021$
   f. None of the other answers

30. The region R is bounded by the graph of the exponential function b^x (where b is a positive constant whose value we forgot to tell you), the x-axis, and the lines $x = 0$ and $x = 1$. When R is revolved about the x-axis one obtains a solid whose volume is 4π-times the area of R. Determine the value of b.



a. $b = 3$

b. $b = 7$

c. $b = \pi$

d. $b = \frac{1+\sqrt{1-2\pi+3\pi^2}}{\pi}$

e. $b = e$

f. None of the other answers

31. What is the smallest amount of area contained in an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,that passes through the point $(1,3)$?

a. $2\pi\sqrt{2}\ \text{ft}^2$

b. $6\pi\ \text{ft}^2$

c. $\pi\sqrt{3}\ \text{ft}^2$

d. $\pi\ \text{ft}^2$

e. $\pi\sqrt{2}\ \text{ft}^2$

f. None of the other answers

32. Suppose the functions $f(x)$ and $g(x)$ are composed to create the function $h(x) = (f \circ g)(x) = 2x + \cos(x)$. Which, if any, of the following statements must be true?

I. $h(x)$ is invertible.
II. The average value of $h(x)$ over the interval $[-\pi, \pi]$ is 0.
III. The graphs of f and g do not have horizontal tangent lines.

a. I. only
b. II. only
c. I. and II. only
d. II. and III. only
e. I., II., and III.
f. None of the other answers

33. Consider the function $f(x) = x^{(x^x)}$. Determine the value of $f'(1)$.

a. $f'(1) = 0$
b. $f'(1) = \ln(2)$
c. $f'(1) = e$
d. $f'(1) = 1$
e. $f(x)$ is not differentiable at $x = 1$.
f. None of the other answers

34. Which of the following expressions represents the area of the region bounded by the graph of $y = x^2$ and its normal line at $(1,1)$?

a. $\int_{-1}^{1} \left(-\frac{x}{2} + \frac{3}{2} - x^2\right) dx$
b. $\int_{-3/2}^{1} \left(-\frac{x}{2} + \frac{3}{2} - x^2\right) dx$
c. $\int_{-1}^{1} (2x - 1 - x^2) dx$
d. $\int_{-3/2}^{1} (2x - 1 - x^2) dx$
e. $\int_{-1}^{1} x^2 dx$
f. None of the other answers