CALCULUS

1. Does $\lim_{x\to 0} \cos\left(\frac{1-\cos x}{x}\right)$ exist? If so, what is the limit?

Answer:____________________

2. A function f is defined on the interval $[a,b]$. Which of the following statements are always true?

   (a) If $f(a) > 0$ and $f(b) < 0$, then there must be a point $c \in (a,b)$ at which $f(c) = 0$.

   (b) If f is continuous on $[a,b]$, $f(a) > 0$ and $f(b) < 0$, then there must be a point $c \in (a,b)$ such that $f(c) = 0$.

   (c) If f is continuous on (a,b), $f(a) > 0$ and $f(b) < 0$, then there must be a point $c \in (a,b)$ such that $f(c) = 0$.

   (d) If f is continuous on $[a,b]$ and there is a point $c \in (a,b)$ such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

Answer:____________________

3. Set

$$f(x) = \begin{cases} ax & x \le 1 \\ bx^2 + x + 1 & x > 1. \end{cases}$$

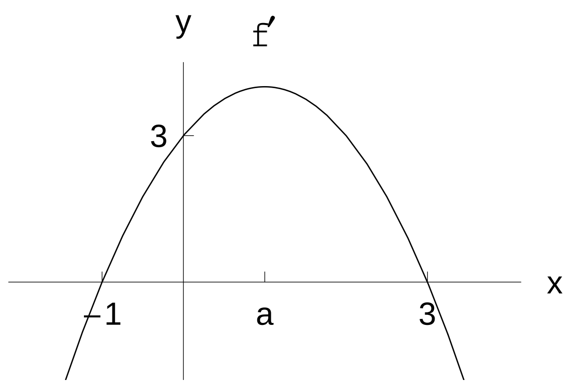
Determine a and b such that f is differentiable at 1.

Answer:____________________

4. Suppose $f(0) = 2$, $f'(0) = 3$, $f(2) = -1$, $f'(2) = -2$, $g(0) = 2$, $g'(0) = 4$, $g(2) = -1$, $g'(2) = 0$. Set $h(x) = f(g(x))$ and find $h'(0)$.

Answer:____________________

5. The graph below is the graph of the derivative of a function f. Given that f is defined and positive on $[-4, 4]$ and $f(0) = 3$, sketch the graph of f, indicating the relative maxima and minima and the points of inflection.



Answer:

6. Find an equation for the normal line to the curve $2x^3 + 2y^3 = 9xy$ at the point $(1, 2)$.

Answer:____________________

7. An athlete is running around a circular track of radius 100 meters at the rate of 5 meters/second. A spectator is 300 meters from the center of the track. How fast is the distance between the runner and the spectator changing when the runner is approaching the spectator and the distance between them is 250 meters?

Answer:____________________

8. A function f is differentiable on an interval (a, b). Which of the following statements are always true?

   (a) If f is nondecreasing on (a, b), then $f'(x) \geq 0$, for all $x \in (a, b)$.

   (b) If f is increasing on (a, b), then $f'(x) > 0$ for all $x \in (a, b)$.

   (c) If f is increasing on (a, b), then $f'(x) > 0$ for at least one $x \in (a, b)$.

   (d) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).

Answer:____________________

9. Given a cone C with radius R and height H. Find the dimensions of the cone with largest volume that can be inscribed in C such that the vertex of the inscribed cone is located at the center of the base of C and the axes of the two cones coincide.

Answer:____________________

10. Given that $\int_1^4 f(x)\,dx = 5,\quad \int_3^4 f(x)\,dx = 7,$ and $\int_1^8 f(x)\,dx = 11,$ find $\int_8^3 f(x)\,dx.$

Answer:____________________

11. Let f be continuous and define F by

$$F(x) = \int_0^x \left[t \int_1^t f(u)\,du \right] dt.$$

Find $F''(1)$.

Answer:____________________

12. The function $f(x) = \int_3^x \sqrt{16+t^2}\,dt$ has an inverse. Find $\left(f^{-1}\right)'(0)$.

Answer:____________________

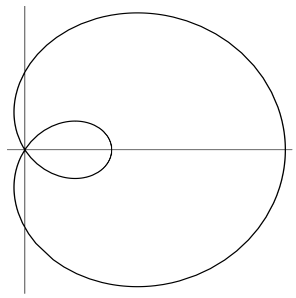
13. Find a linear polynomial P such that $P(0)=1$ and $\int \frac{P(x)}{x^2(x-1)^2}$ is a rational function.

Answer:____________________

14. The region bounded by the graph of $y = e^{-x^2}$ and the x-axis, $0 \le x \le 1$ is rotated around the y-axis. Find the volume of the solid that is generated.

Answer:____________________

15. The curve $r = 1 + 2\cos\theta$ is shown in the figure. Find the area of the inner loop of the curve.

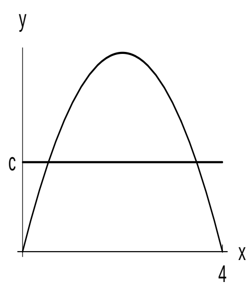


Answer:____________________

16. A curve in the plane is defined by the parametric equations: $x = 3t^2 + 1,\ y = \frac{2}{3}t^3,\ 0 \le t \le 4.$ Find the length of the curve.

Answer:____________________

17. Find the number c such that the line $y = c$ divides the region bounded by $y = 4x - x^2$ and the x-axis into two sub-regions of equal area.



Answer:____________________

18. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are always true?

(a) If $a_n > 0$ for all n and $a_n \to L$, then $L > 0$.

(b) If $a_n \geq 0$ for all n and $a_n \to L$, then $L \geq 0$.

(c) If $\{a_n\}$ is not bounded, then it diverges.

(d) If $\{a_n\}$ is decreasing, then it converges.

Answer:____________________

19. Find all values p such that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,(\ln\, k)^p}$ converges.

Answer:____________________

20. Let f be a function such that $|\,f^{(n)}(x)| \leq 3$ for all x and n. Find the least integer n such that the Taylor polynomial of degree n at $x = 0$ approximates $f(1/2)$ to within 0.0005.

Answer:____________________

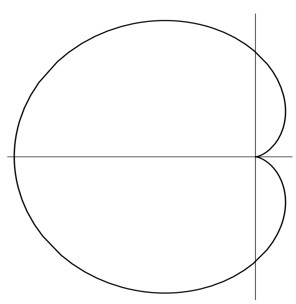
21. Let $f(x) = x^2 \sin x$. Find $f^{(9)}(0)$. HINT: Use Taylor series.

Answer:____________________

22. At each point (x, y) on a certain curve, the slope of the curve is $3x^2y$. Find an equation for the curve given that it passes through the point $(0, 8)$.

Answer:____________________

23. Find the rectangular coordinates of the points on the cardiod $r = 1 - \cos\theta$ at which the tangent line to the curve is vertical. HINT: Parametrize the curve.



Answer:____________________

24. Evaluate the improper integral $\int_0^\infty \frac{1}{e^x + e^{-x}}\, dx$.

Answer:____________________

25. An advertising company has designed a campaign to introduce a new product to city of 50 thousand people. Let $P = P(t)$ denote the number of people are aware of the product at time t and assume that P increases at a rate proportional to the number of people still unaware of the product. If no one knew about the product at the beginning of the campaign $[P(0) = 0]$ and 30% of the people are aware of the product after 10 days of advertising, how long will it take for 90% of the population to be aware of the product?

Answer:____________________