

CALCULUS

1. Find a number a , if possible, such that $\lim_{x \rightarrow 3} \frac{2x^2 - 3ax + x - a - 1}{x^2 - 2x - 3}$ exists.

Answer: $a = 2$

2. Let f be some function for which you know only that

$$\text{if } 0 < |x - 3| < 1, \quad \text{then } |f(x) - 5| < 0.1.$$

Which of the following statements are necessarily true?

- (a) $\lim_{x \rightarrow 3} f(x) = 5$.
- (b) If $|x - 3| < 1$, then $|f(x) - 5| < 0.1$.
- (c) If $|x - 2.5| < 0.3$, then $|f(x) - 5| < 0.1$.
- (d) If $0 < |x - 3| < 0.5$, then $|f(x) - 5| < 0.1$.
- (e) If $|x - 3| < \frac{1}{4}$, then $|f(x) - 5| < \frac{1}{4}(0.1)$.
- (f) If $\lim_{x \rightarrow 3} f(x) = L$, then $4.9 \leq L \leq 5.1$.

Answer: (c), (d), (f)

3. Which of the following statements are always true?

- (a) If $f(1) < 0$ and $f(2) > 0$, then there must be a point $c \in (1, 2)$ such that $f(c) = 0$.
- (b) If f is continuous on $[1, 2]$, $f(1) < 0$ and $f(2) > 0$, then there must be a point $c \in (1, 2)$ such that $f(c) = 0$.
- (c) If f is continuous on $[1, 2]$ and there is a point c in $(1, 2)$ such that $f(c) = 0$, then $f(1)$ and $f(2)$ have opposite sign.
- (d) If f has no zeros and is continuous on $[1, 2]$, then $f(1)$ and $f(2)$ have the same sign.

Answer: (b), (d)

4. Set

$$f(x) = \begin{cases} 2x^2 - 1 & x < 2, \\ A & x = 2, \\ x^3 + Bx + C & x > 2. \end{cases}$$

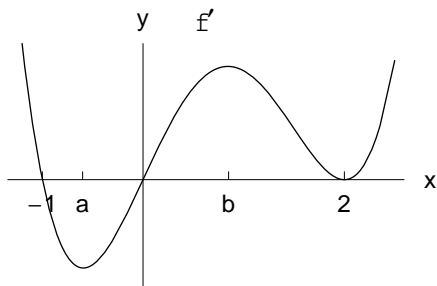
Determine A , B and C such that f is differentiable at 2.

Answer: $A = 7$, $B = -4$, $C = 7$

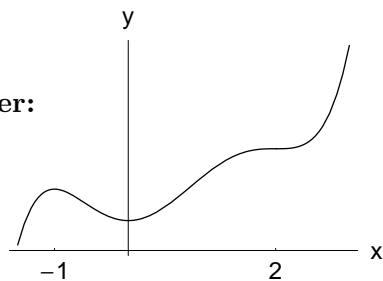
5. Suppose $f'(2) = 4$, $g'(2) = -3$, $f(2) = -5$, $g(2) = 3$, $f'(3) = -3$. Set $h(x) = f(g(x))$ and find $h'(2)$.

Answer: 9

6. The graph below is the graph of the derivative of a function f . Given that $f(0) = 1$, sketch the graph of f , indicating the relative maxima and minima and the points of inflection.



Answer:



7. Find an equation for the normal line to the curve $x^2 + xy - 2y^2 = 4$ at the point $(3, -1)$.

Answer: $y + 1 = \frac{7}{5}(x - 3)$

8. A conical paper cup 8 inches across the top and 6 inches deep is full of water. The cup springs a leak at the bottom and loses water at the rate of 2 cubic inches per minute. How fast is the water level in the cup falling at the instant the water is 3 inches deep?

Answer: $-\frac{1}{2\pi}$ in/min

9. A piece of wire 100 inches long is to be cut into pieces and used to construct the skeleton of a box with a square base. Find the maximum volume of the box.

Answer: $V = \left(\frac{25}{3}\right)^3 \approx 578.704$

10. Given that $xy + x^2 - y^2 = 1$, find $\frac{d^2y}{dx^2}$ at the point $(1, 1)$.

Answer: $y'' = -10$

11. Given that $\int_0^1 f(x) dx = \frac{4}{3}$, $\int_1^2 f(x) dx = \frac{8}{3}$, and $\int_0^3 f(x) dx = \frac{11}{3}$, find $\int_2^3 f(x) dx$.

Answer: $-\frac{1}{3}$

12. Suppose that $\int_c^x f(t) dt = 5x^3 + 40$. Find f and c .

Answer: $f(x) = 15x^2$, $c = -2$

13. The function $f(x) = \int_4^{x^2} \sqrt{9+t^2} dt$ has an inverse. Find $(f^{-1})'(0)$.

Answer: $(f^{-1})'(0) = \frac{1}{4\sqrt{13}}$

14. Evaluate the integral $\int_0^2 f(x) dx$ where f is the function whose graph is

Answer: 1

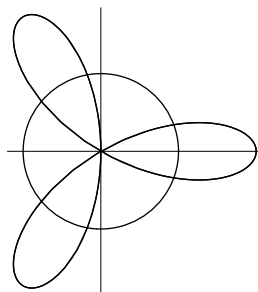
15. Calculate the derivative of $G(x) = \int_0^x \frac{1}{1+t^2} dt + \int_0^{1/x} \frac{1}{1+t^2} dt$. What can you conclude about the function G ?

Answer: $G'(x) = 0$; $G(x) = C$ (constant)

16. The region bounded by the graph of $y = \cos x$ and the x -axis, $0 \leq x \leq \pi/2$ is rotated around the y -axis. Find the volume of the solid that is generated.

Answer: $2\pi \left[\frac{\pi}{2} - 1\right] \approx 3.586$

17. The curves $r = 2 \cos 3\theta$ and $r = 1$ are shown in the figure. Find the area of the shaded region.



Answer: $\frac{\pi}{9} + \frac{\sqrt{3}}{6} \approx 0.638$

18. A right triangle with hypotenuse of length a is rotated about one of its legs to form a right circular cone. Find the greatest possible volume of such a cone.

Answer: $\frac{2\pi\sqrt{3}a^3}{27}$

19. Let $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}$ on the interval $[0, 2]$. Find the length of the graph of f .

Answer: $\frac{14}{3}$

20. Arrange the functions $f_1(x) = x^e$, $f_2(x) = e^x$, $f_3(x) = x^x$, $f_4(x) = 2^x$, $f_5(x) = (\ln x)^{2x}$ in increasing order of growth.

Answer: x^e , 2^x , e^x , $(\ln x)^{2x}$, x^x

21. Let $\{a_n\}$ be the sequence defined by

$$a_n = \begin{cases} n^2/(n^2 - 10) & \text{if } n \text{ is a multiple of } 3, \\ n/(n + 1) & \text{if } n \text{ has the form } n = 3k + 1, \\ \sqrt{n}/\sqrt{4n + 5} & \text{if } n \text{ has the form } n = 3k + 2. \end{cases}$$

Determine whether or not $\lim_{n \rightarrow \infty} a_n$ exists. If it does, give the limit.

Answer: The limit does not exist.

22. A ball rebounds to two-thirds of the height from which it falls. If it is dropped from a height of 6 feet and is allowed to continue bouncing indefinitely, what is the total distance it travels?

Answer: 30 ft.

23. Let $\{a_n\}$ be the sequence defined by

$$a_n = \frac{n!}{n^n}.$$

Determine whether or not $\lim_{n \rightarrow \infty} a_n$ exists. If it does, give the limit.

Answer: The sequence converges to 0.

24. Evaluate $\lim_{x \rightarrow \infty} e^{-x^2} \int_0^x e^{t^2} dt$

Answer: 0

25. A curve in the plane has the property that a normal line to the curve at each point $P(x, y)$ always passes through the point $(2, 0)$. Find an equation for the curve given that it passes through the point $(1, 1)$.

Answer: $(x - 2)^2 + y^2 = 2$

26. Two particles start at the same instant, the first along the ray

$$x(t) = 2t + 6, \quad y(t) = 5 - 4t, \quad t \geq 0,$$

and the second along the circular path

$$x(t) = 3 - 5 \cos \pi t, \quad y(t) = 1 + 5 \sin \pi t, \quad t \geq 0.$$

At what points, if any, do the paths intersect? At what points, if any, do the particles collide?

Answer: Intersect at $(6, 5)$; collide at $(8, 1)$.

27. Evaluate the improper integral $\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx$.

Answer: $\frac{\pi}{4}$

28. Evaluate the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{n}{n}\right)^3 \right].$$

Hint: Recognize the expression as a Riemann sum and evaluate the associated definite integral.

Answer: $\frac{1}{4}$

29. Points A and B are diametrically opposite each other on a lake of radius 1 mile. A man wishing to go from A to B can either walk around the circumference of the lake, or row directly to B , or row to a point C on the circumference and then walk to B . If he can row at 2 miles per hour and walk at 4 miles per hour, which of the three routes will take minimum time?

Answer: Walk around the lake.

30. Oil is being pumped continuously from an Alaskan oil well at a rate proportional to the amount of oil left in the well. That is, $\frac{d}{dt}y(t) = ky(t)$ where $y(t)$ is the amount of oil left in the well at time t . There were 1,000,000 barrels of oil in the well initially and 500,000 barrels 6 years later. Assume that it is not profitable to pump oil when there are fewer than 50,000 barrels remaining. How long will it be profitable to pump oil?

Answer: $t = -\frac{6 \ln(0.05)}{\ln 2} \approx 25.93$ years