CALCULUS

1. Find a number $a$, if possible, such that $\lim_{x \to 3} \frac{2x^2 - 3ax + x - a - 1}{x^2 - 2x - 3}$ exists.

**Answer:** $a = 2$

2. Let $f$ be some function for which you know only that

$$\text{if } \ 0 < |x - 3| < 1, \text{ then } |f(x) - 5| < 0.1.$$ 

Which of the following statements are necessarily true?

(a) $\lim_{x \to 3} f(x) = 5$.

(b) If $|x - 3| < 1$, then $|f(x) - 5| < 0.1$.

(c) If $|x - 2.5| < 0.3$, then $|f(x) - 5| < 0.1$.

(d) If $0 < |x - 3| < 0.5$, then $|f(x) - 5| < 0.1$.

(e) If $|x - 3| < \frac{1}{4}$, then $|f(x) - 5| < \frac{1}{4}(0.1)$.

(f) If $\lim_{x \to 3} f(x) = L$, then $4.9 \leq L \leq 5.1$.

**Answer:** (c), (d), (f)

3. Which of the following statements are always true?

(a) If $f(1) < 0$ and $f(2) > 0$, then there must be a point $c \in (1, 2)$ such that $f(c) = 0$.

(b) If $f$ is continuous on $[1, 2]$, $f(1) < 0$ and $f(2) > 0$, then there must be a point $c \in (1, 2)$ such that $f(c) = 0$.

(c) If $f$ is continuous on $[1, 2]$ and there is a point $c$ in $(1, 2)$ such that $f(c) = 0$, then $f(1)$ and $f(2)$ have opposite sign.

(d) If $f$ has no zeros and is continuous on $[1, 2]$, then $f(1)$ and $f(2)$ have the same sign.

**Answer:** (b), (d)
4. Set
\[ f(x) = \begin{cases} 
2x^2 - 1 & x < 2, \\
A & x = 2, \\
x^3 + Bx + C & x > 2. 
\end{cases} \]

Determine \( A, B \) and \( C \) such that \( f \) is differentiable at \( 2 \).

**Answer:** \( A = 7, \ B = -4, \ C = 7 \)

5. Suppose \( f'(2) = 4, \ g'(2) = -3, \ f(2) = -5, \ g(2) = 3, \ f'(3) = -3 \). Set \( h(x) = f(g(x)) \) and find \( h'(2) \).

**Answer:** 9

6. The graph below is the graph of the derivative of a function \( f \). Given that \( f(0) = 1 \), sketch the graph of \( f \), indicating the relative maxima and minima and the points of inflection.

![Graph of f'(x)](image)

**Answer:**

7. Find an equation for the normal line to the curve \( x^2 + xy - 2y^2 = 4 \) at the point \( (3, -1) \).

**Answer:** \( y + 1 = \frac{7}{3}(x - 3) \)
8. A conical paper cup 8 inches across the top and 6 inches deep is full of water. The cup springs a leak at the bottom and loses water at the rate of 2 cubic inches per minute. How fast is the water level in the cup falling at the instant the water is 3 inches deep?

**Answer:** \(-\frac{1}{2\pi}\) in/min

9. A piece of wire 100 inches long is to be cut into pieces and used to construct the skeleton of a box with a square base. Find the maximum volume of the box.

**Answer:** \(V = \left(\frac{25}{3}\right)^3 \approx 578.704\)

10. Given that \(xy + x^2 - y^2 = 1\), find \(\frac{d^2y}{dx^2}\) at the point \((1, 1)\).

**Answer:** \(y'' = -10\)

11. Given that \(\int_0^1 f(x)\,dx = \frac{4}{3}\), \(\int_1^2 f(x)\,dx = \frac{8}{3}\), and \(\int_0^3 f(x)\,dx = \frac{11}{3}\), find \(\int_2^3 f(x)\,dx\).

**Answer:** \(-\frac{1}{3}\)

12. Suppose that \(\int_c^x f(t)\,dt = 5x^3 + 40\). Find \(f\) and \(c\).

**Answer:** \(f(x) = 15x^2\), \(c = -2\)

13. The function \(f(x) = \int_4^{x^2} \sqrt{9 + t^2}\,dt\) has an inverse. Find \((f^{-1})'(0)\).

**Answer:** \((f^{-1})'(0) = \frac{1}{4\sqrt{13}}\)

14. Evaluate the integral \(\int_0^2 f(x)\,dx\) where \(f\) is the function whose graph is

**Answer:** 1
15. Calculate the derivative of \( G(x) = \int_0^x \frac{1}{1+t^2} \, dt + \int_0^{1/x} \frac{1}{1+t^2} \, dt \). What can you conclude about the function \( G \)?

**Answer:** \( G'(x) = 0 \); \( G(x) = C \) (constant)

16. The region bounded by the graph of \( y = \cos x \) and the \( x \)-axis, \( 0 \leq x \leq \pi/2 \) is rotated around the \( y \)-axis. Find the volume of the solid that is generated.

**Answer:** \( 2\pi \left[ \frac{\pi}{2} - 1 \right] \approx 3.586 \)

17. The curves \( r = 2 \cos 3\theta \) and \( r = 1 \) are shown in the figure. Find the area of the shaded region.

![Figure](image)

**Answer:** \( \frac{\pi}{9} + \frac{\sqrt{3}}{6} \approx 0.638 \)

18. A right triangle with hypotenuse of length \( a \) is rotated about one of its legs to form a right circular cone. Find the greatest possible volume of such a cone.

**Answer:** \( \frac{2\sqrt{3}a^3}{27} \)

19. Let \( f(x) = \frac{1}{3} (x^2 + 2)^{3/2} \) on the interval \([0, 2]\). Find the length of the graph of \( f \).

**Answer:** \( \frac{14}{3} \)

20. Arrange the functions \( f_1(x) = x^e, f_2(x) = e^x, f_3(x) = x^x, f_4(x) = 2^x, f_5(x) = (\ln x)^{2x} \) in increasing order of growth.

**Answer:** \( x^e, 2^x, e^x, (\ln x)^{2x}, x^x \)
21. Let \( \{a_n\} \) be the sequence defined by

\[
a_n = \begin{cases} 
\frac{n^2}{(n^2 - 10)} & \text{if } n \text{ is a multiple of } 3, \\
\frac{n}{n+1} & \text{if } n \text{ has the form } n = 3k + 1, \\
\sqrt{n}/\sqrt{4n+5} & \text{if } n \text{ has the form } n = 3k + 2.
\end{cases}
\]

Determine whether or not \( \lim_{n \to \infty} a_n \) exists. If it does, give the limit.

**Answer:** The limit does not exist.

22. A ball rebounds to two-thirds of the height from which it falls. If it is dropped from a height of 6 feet and is allowed to continue bouncing indefinitely, what is the total distance it travels?

**Answer:** 30 ft.

23. Let \( \{a_n\} \) be the sequence defined by

\[a_n = \frac{n!}{n^n}.
\]

Determine whether or not \( \lim_{n \to \infty} a_n \) exists. If it does, give the limit.

**Answer:** The sequence converges to 0.

24. Evaluate \( \lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2} \, dt \)

**Answer:** 0

25. A curve in the plane has the property that a normal line to the curve at each point \( P(x, y) \) always passes through the point \( (2, 0) \). Find an equation for the curve given that it passes through the point \( (1, 1) \).

**Answer:** \( (x - 2)^2 + y^2 = 2 \)

26. Two particles start at the same instant, the first along the ray

\[x(t) = 2t + 6, \quad y(t) = 5 - 4t, \quad t \geq 0,\]

and the second along the circular path

\[x(t) = 3 - 5 \cos \pi t, \quad y(t) = 1 + 5 \sin \pi t, \quad t \geq 0.\]
At what points, if any, do the paths intersect? At what points, if any, do the particles collide?

**Answer:** Intersect at (6, 5); collide at (8, 1).

27. Evaluate the improper integral \( \int_0^\infty \frac{1}{e^x + e^{-x}} \, dx \).

**Answer:** \( \frac{\pi}{4} \)

28. Evaluate the limit

\[
\lim_{n \to \infty} \frac{1}{n} \left[ \left( \frac{1}{n} \right)^3 + \left( \frac{2}{n} \right)^3 + \cdots + \left( \frac{n}{n} \right)^3 \right].
\]

Hint: Recognize the expression as a Riemann sum and evaluate the associated definite integral.

**Answer:** \( \frac{1}{4} \)

29. Points \( A \) and \( B \) are diametrically opposite each other on a lake of radius 1 mile. A man wishing to go from \( A \) to \( B \) can either walk around the circumference of the lake, or row directly to \( B \), or row to a point \( C \) on the circumference and then walk to \( B \). If he can row at 2 miles per hour and walk at 4 miles per hour, which of the three routes will take minimum time?

**Answer:** Walk around the lake.

30. Oil is being pumped continuously from an Alaskan oil well at a rate proportional to the amount of oil left in the well. That is, \( \frac{dy}{dt} = ky(t) \) where \( y(t) \) is the amount of oil left in the well at time \( t \). There were 1,000,000 barrels of oil in the well initially and 500,000 barrels 6 years later. Assume that it is not profitable to pump oil when there are fewer than 50,000 barrels remaining. How long will it be profitable to pump oil?

**Answer:** \( t = -\frac{6 \ln(0.05)}{\ln 2} \approx 25.93 \) years