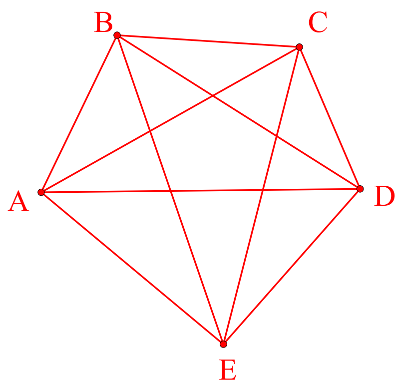
Name____

University of Houston
High School Mathematics Contest
2005 Geometry Exam - Key

Directions:
You have 50 minutes to complete this exam. Calculators are not permitted. You must write your answer in the answer blank provided. Answers should be exact (such as $\frac{16\pi}{5} - 2\sqrt{3}$) and should be written in simplest form. If units are given in the question, then the answer should be written with appropriate units.

1. How many lines are determined by five coplanar points, provided that no three of them are collinear?

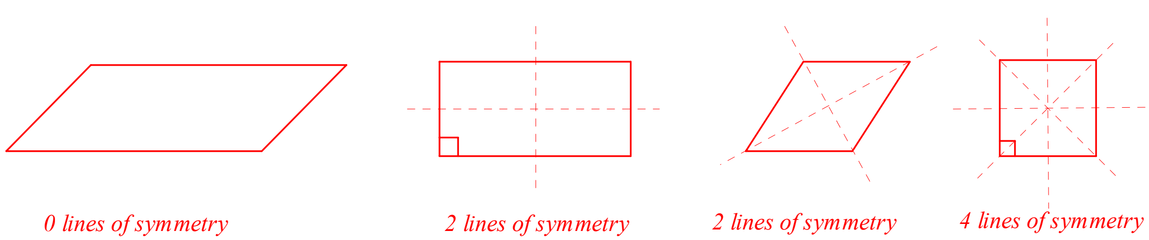


_______10_________

For simplicity in the diagram, segments have been drawn between any two points rather than lines.

2. A parallelogram can have at most _____ lines of symmetry. (Be sure to answer in the large blank below.)

________4_________



A square is a parallelogram and has 4 lines of symmetry.

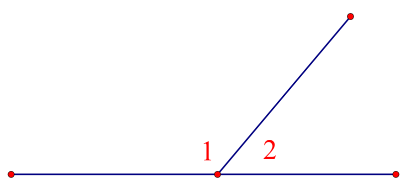
3. Write the converse of the following statement and then state whether the converse is true or false:
If two angles form a linear pair, then they are supplementary angles.

Converse:
If two angles are supplementary angles, then they form a linear pair.
(Note: The words in parentheses above represent slightly different correct wording; just make sure that the order is correct and the noun in the hypothesis is clear -- Students should NOT say, for example, "If ***they*** *are supplementary angles, then two angles form a linear pair.")*

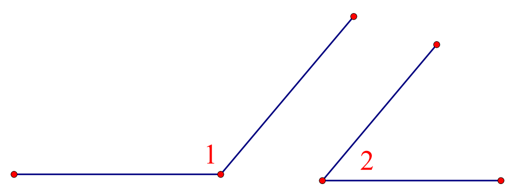
Explanation: If the conditional statement is "If p, then q" the converse is "If q, then p." (The statement should be adjusted as necessary to ensure clarity and proper grammar.)

Is the converse true or false? ___**False**_______
(Both answers must be correct in order for the student to receive credit for this question.)

Explanation: Two angles may be supplementary but not adjacent, in which case they do not form a linear pair. See the figures below.



Angles 1 and 2 are supplementary and form a linear pair.



Angles 1 and 2 are supplementary but do not form a linear pair.

4. If $M(3,-5)$ is the midpoint of the line segment joining points A and B, and A has coordinates $(-1,-2)$, find the coordinates of B.

____$(7,-8)$_______

Given two points $A(x_1, y_1)$ *and* $B(x_2, y_2)$, *the coordinates of the midpoint M are* $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. *Notice that the coordinates of the* *<u>midpoint</u>* *and one endpoint are already given in this question (rather than the coordinates of both endpoints).*

If we let (x, y) *represent the coordinates of point B, then the midpoint of the segment joining point* $A(-1,-2)$ *and* $B(x, y)$ *can be represented by* $M\left(\frac{-1+x}{2}, \frac{-2+y}{2}\right)$. *Since we already know that the coordinates of point M are* $(3,-5)$, *we can set up the equations* $\frac{-1+x}{2}=3$ *and* $\frac{-2+y}{2}=-5$. *Solving these equations, we obtain the following:*

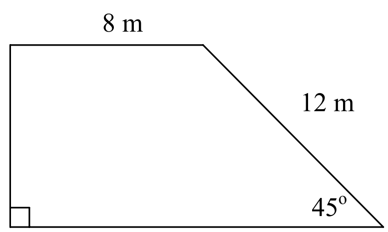
$$\frac{-1+x}{2}=3 \quad \Rightarrow \quad -1+x=6 \quad \Rightarrow \quad x=7$$

$$\frac{-2+y}{2}=-5 \quad \Rightarrow \quad -2+y=-10 \quad \Rightarrow \quad y=-8$$

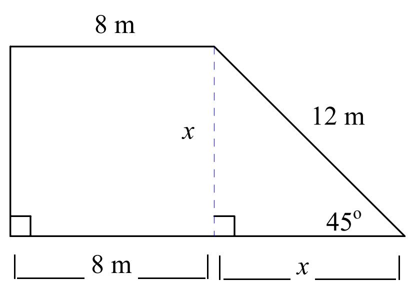
Therefore, the coordinates of point B are $(7,-8)$.

5. Find the area of the following trapezoid:

__$\left(48\sqrt{2}+36\right)$ m^2__



First, we draw an altitude of the trapezoid as shown below, which divides the trapezoid into a rectangle and a triangle:



In a 45°-45°-90° triangle, the length of the hypotenuse is $\sqrt{2}$ *times the length of a leg, so* $x\sqrt{2}=12$. *Therefore,*

$$x=\frac{12}{\sqrt{2}}=\frac{12}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{12\sqrt{2}}{2}=6\sqrt{2}.$$

The area of the trapezoid is equal to the sum of the area of the rectangle and the area of the trapezoid.

Area of rectangle $= 8x=8\left(6\sqrt{2}\right)=48\sqrt{2}\ \text{m}^2$

Area of triangle $= \frac{1}{2}x^2=\frac{1}{2}\left(6\sqrt{2}\right)^2=\frac{1}{2}(36)(2)=36\ \text{m}^2$

Therefore, the area of the trapezoid is $\left(48\sqrt{2}+36\right)$ m^2.

6. The supplement of an angle is 22 degrees less than three times its complement. Find the measure of the angle.

_______34°_________

34(with no degree symbol) is also fine; many high school Geometry books assume degree measure if no unit is stated.

Let $x =$ *the angle*
Then $180 - x =$ *the supplement of the angle*
Then $90 - x =$ *the complement of the angle*

Equation:

$$180 - x = 3(90 - x) - 22$$

$$180 - x = 270 - 3x - 22$$

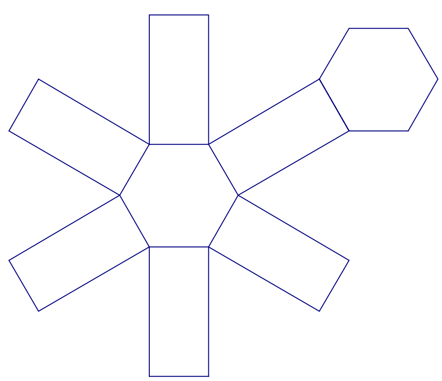
$$2x = 68$$

$$x = 34$$

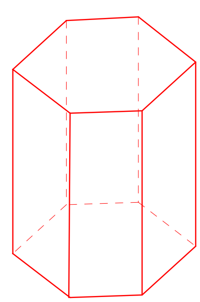
Therefore, the measure of the angle is 34°.

7. Below is a net that can be used to create a solid, formed by six congruent rectangles and two congruent regular hexagons. How many edges does the solid have?

________18________

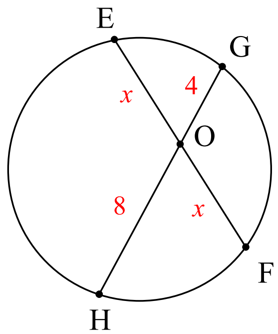


When folded into a solid, this net would form a regular right hexagonal prism, as shown below. The prism has six edges along the top base, six edges along the bottom base, and six lateral edges (the vertical edges in the diagram below) – for a total of 18 edges.



8. In the circle below, chord $\overline{GH}$ bisects chord $\overline{EF}$ at point *O*. If the length of $\overline{GO}$ is 4 cm and the length of $\overline{OH}$ is 8 cm, find the length of chord $\overline{EF}$.

____$8\sqrt{2}$ cm_______



Since chord $\overline{GH}$ bisects chord $\overline{EF}$ at point O, $\overline{GH} \cong \overline{EF}$. In the diagram, both $\overline{GH}$ and $\overline{EF}$ have been labeled as x. If two chords intersect within a circle, then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.

$x \cdot x = 4 \cdot 8$

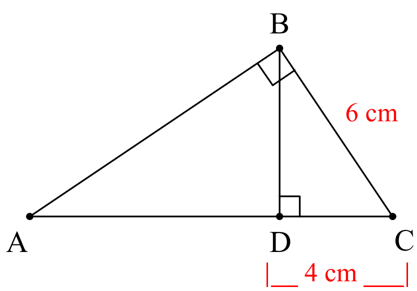
$x^2 = 32$

$x = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

$EF = 2x$, so the length of chord $\overline{EF}$ is $8\sqrt{2}$ cm.

9. Altitude $\overline{BD}$ is drawn to hypotenuse $\overline{AC}$ of right triangle *ABC*. If the length of $\overline{DC}$ is 4 cm and the length of $\overline{BC}$ is 6 cm, find the length of $\overline{AD}$.

_____5 cm_________

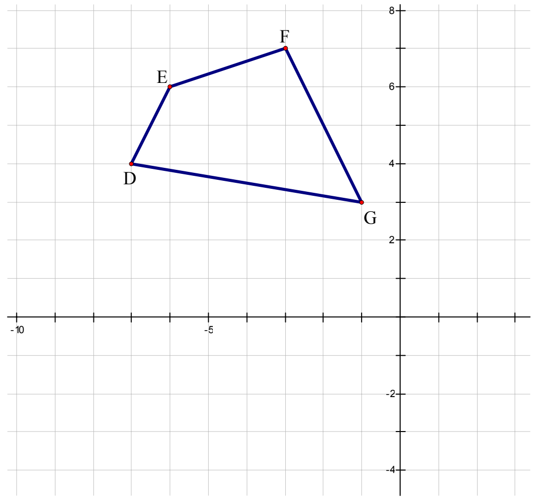


The altitude to the hypotenuse of a right triangle forms two triangles that are similar to the original triangle and to each other. This leads to the following corollary:
The altitude to the hypotenuse of a right triangle intersects the hypotenuse so that the length of each leg is the geometric mean between the length of the adjacent segment of the hypotenuse and the length of the entire hypotenuse.

We can therefore state that $\frac{DC}{BC} = \frac{BC}{AC}$. (This same ratio could be obtained by drawing ΔABC and ΔBDC separately and setting up a proportion using corresponding sides of these similar triangles.)

Since $\frac{DC}{BC} = \frac{BC}{AC}$, we can substitute the known values and find that $\frac{4}{6} = \frac{6}{AC}$.
Cross-multiplying, we obtain $4AC = 36$, so $AC = 9$. Since $AC = 9$ and $DC = 4$, $AD = 9 - 4 = 5$. Therefore, the length of $\overline{AD}$ is 5 cm.

10. If quadrilateral $DEFG$ is rotated 90° counterclockwise with center of rotation $(-2,1)$, give the coordinates of each of the vertices of the image $D'E'F'G'$.



D': __$(-5,-4)$____

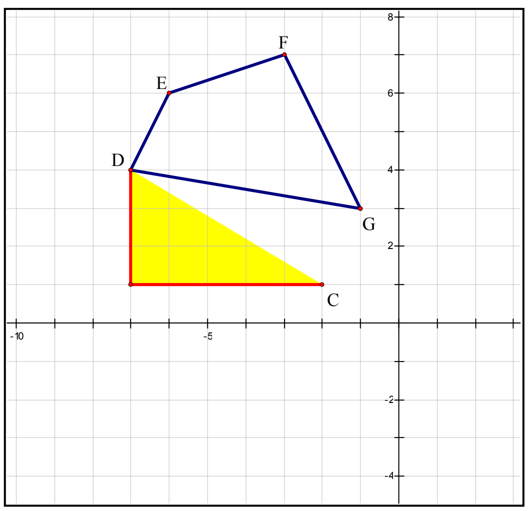
E': __$(-7,-3)$____

F': __$(-8,0)$_____

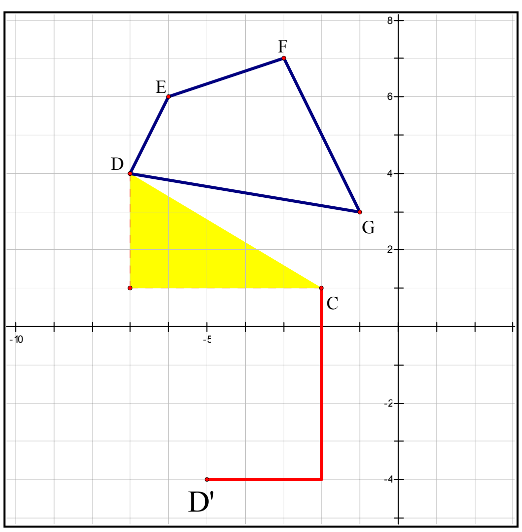
G': __$(-4,2)$_____

(All coordinates above must be correct in order for the student to receive credit for this question.)

There are a variety of ways to approach this problem, one of which is shown below.



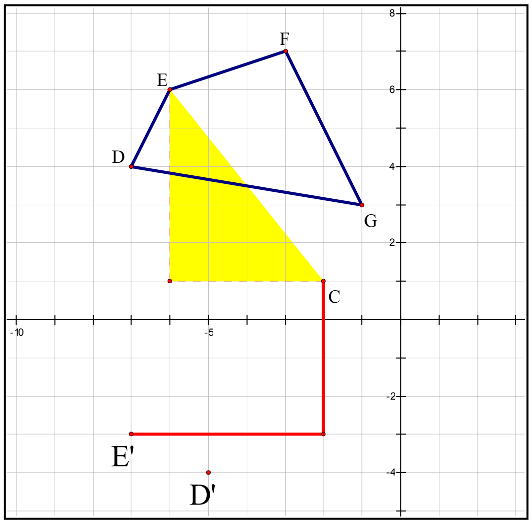
It is necessary to map each vertex of the quadrilateral to its new image point, so each one will be rotated individually around center of rotation $(-2,1)$. The rotation of point D will be described in detail, and the same method will be used for rotating points E, F, and G. If we call the center of rotation C, then let $\overline{CD}$ represent the hypotenuse of a right triangle, as shown in the figure below. We now want to focus on the legs of this right triangle.



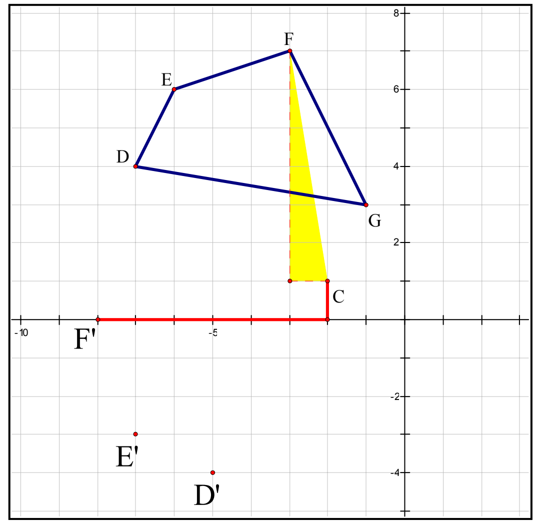
Look at the horizontal and vertical legs of the right triangle shown at the above left. Picture them like an "L" (reversed in this case). The "L" shape initially goes 5 units to the left and 3 units up from point C. Rotating this "L" counterclockwise about center of rotation $(-2,1)$, we then draw an "L" shape that goes 5 units down and 3 units to the left (as shown in the figure at the left). We then label the image point D'.

We then repeat the same procedure to find the images of E, F, and G. A diagram of each is shown below, along with the final solution.

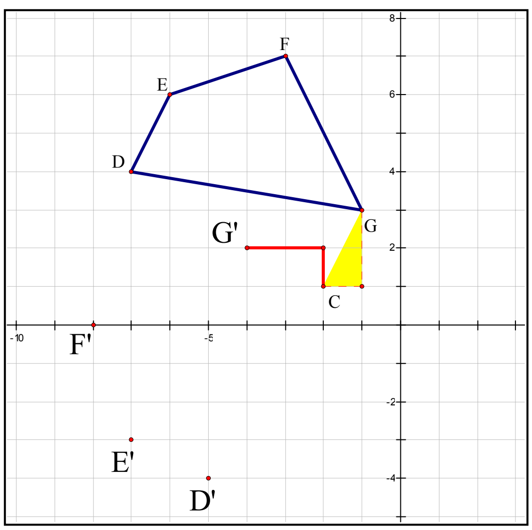
Rotating point E to find the location of E':



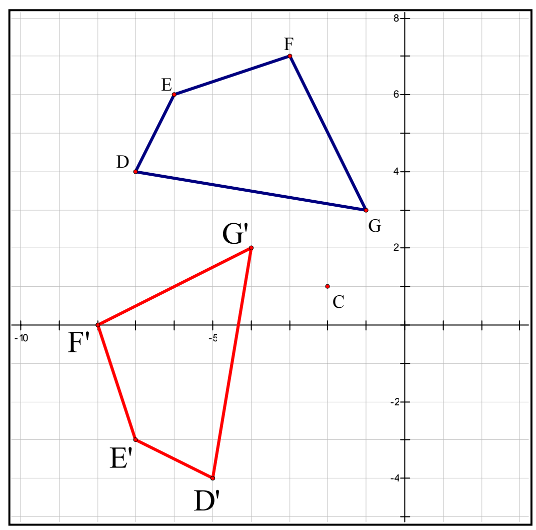
Rotating point F to find the location of F':



Rotating point G to find the location of G':



FINAL SOLUTION



As seen in the final solution above, the coordinates of image $D'E'F'G'$ *are as follows:*

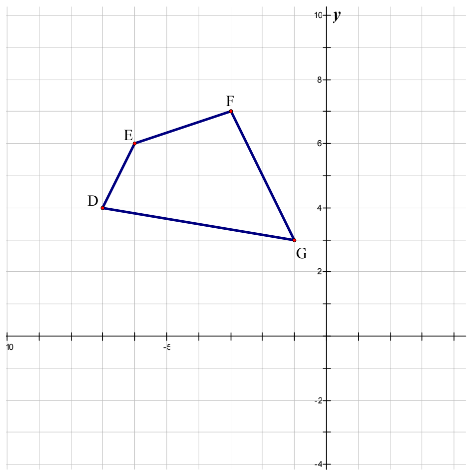
D': $(-5,-4)$

E': $(-7,-3)$

F': $(-8,0)$

G': $(-4,2)$

11. If quadrilateral *DEFG* is dilated with a scale factor of 2 using center of dilation $(-5, 9)$, give the coordinates of each of the vertices of the image $D'E'F'G'$.



D': __$(-9, -1)$____

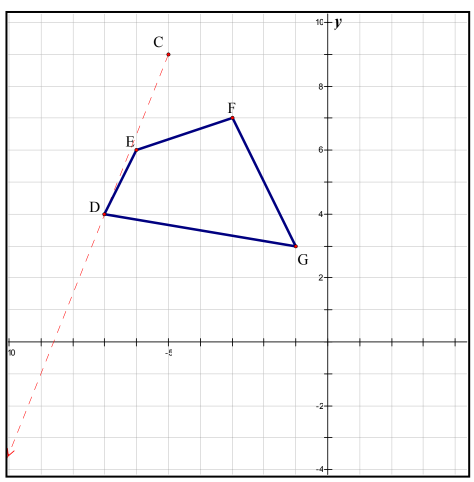
E': ___$(-7, 3)$____

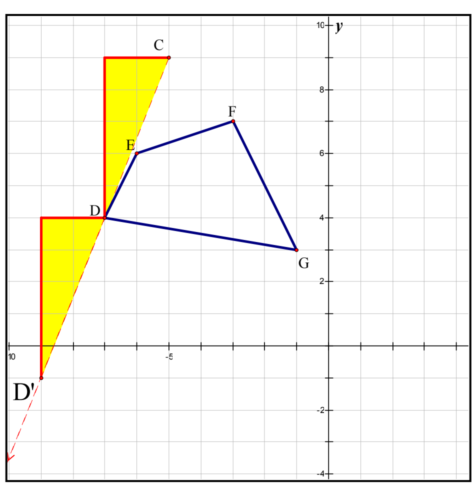
F': ___$(-1, 5)$____

G': ___$(3, -3)$____

(All coordinates above must be correct in order for the student to receive credit for this question.)

There are a variety of ways to approach this problem, one of which is shown below.



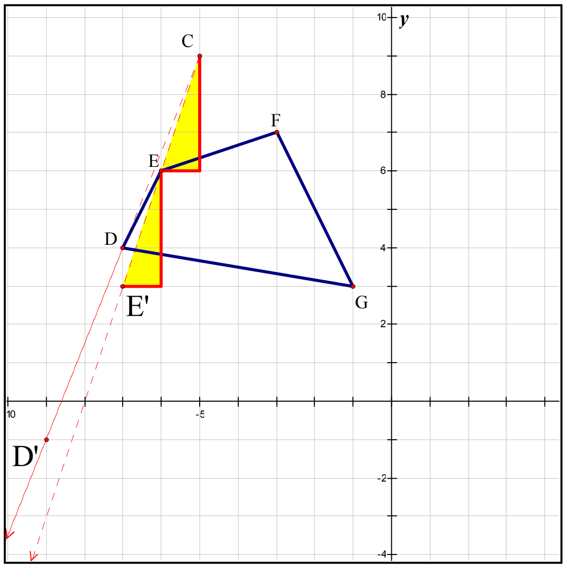


It is necessary to map each vertex of the quadrilateral to its new image point, so each one will be dilated individually about center of dilation $(-5, 9)$. The dilation of point D will be described in detail, and the same method will be used for dilating points E, F, and G. Let us call the center of dilation C. e first draw a ray that begins at point C and passes through point D. Look at segment $\overline{CD}$. Since we want to dilate point D with a scale factor of 2, we want to place point D' on $\overrightarrow{CD}$ in such a way that $CD' = 2CD$. (In other words, we want point D' to be twice as far away from point C as was point D.)

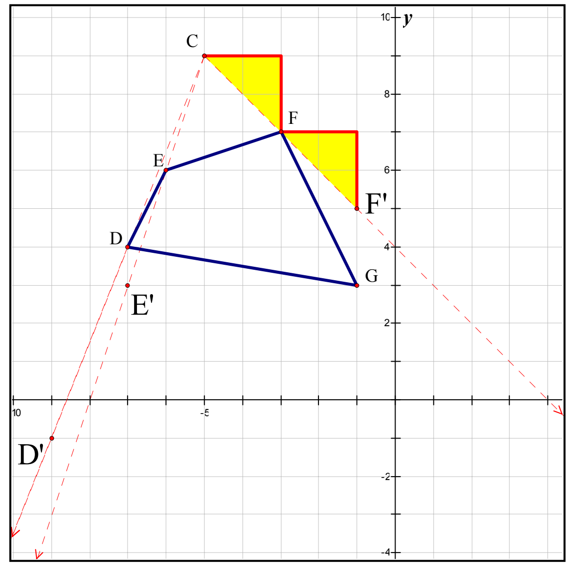
Let $\overline{CD}$ represent the hypotenuse of a right triangle, as shown in the figure at the left. We now want to focus on the legs of this ight triangle. To get from point C to point D, we can go 2 units to the left and 5 units down. To go twice as far away from point C, we want to repeat this process one more time. From point D, go 2 units to the left and 5 units down, and label this point D', as shown in the figure at the left.

We then repeat the same procedure to find the images of E, F, and G. A diagram of each is shown below, along with the final solution.

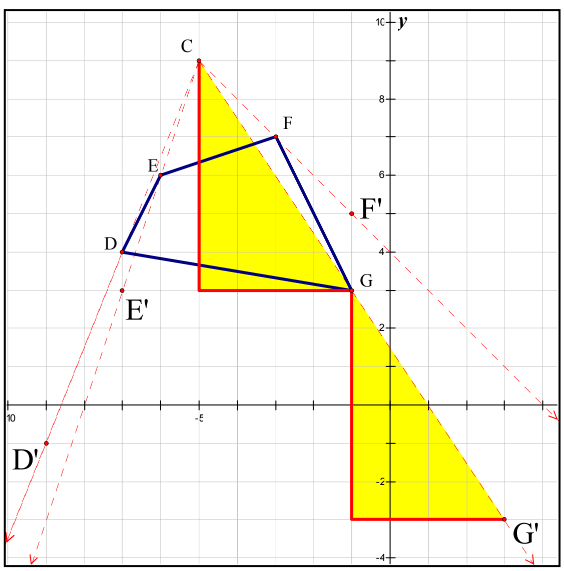
Dilating point E to find the location of E':



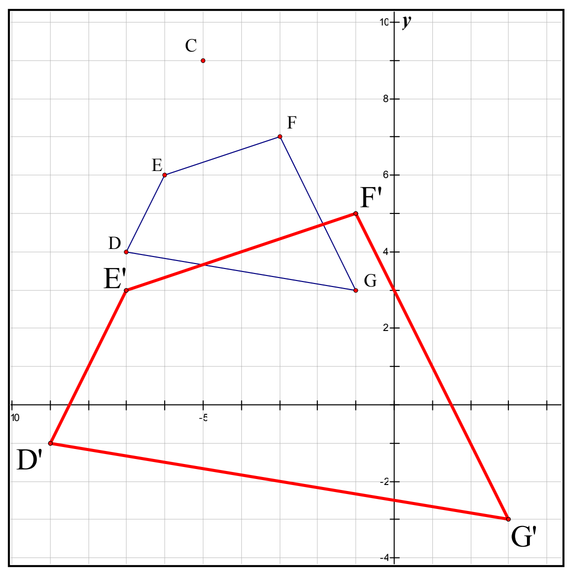
Dilating point F to find the location of F':



Dilating point G to find the location of G':



FINAL SOLUTION



As seen in the final solution above, the coordinates of image $D'E'F'G'$ are as follows:

D': $(-9,-1)$

E': $(-7,3)$

F': $(-1,5)$

G': $(3,-3)$

12. Pyramid A is similar to pyramid B. If pyramid A has a surface area of 400 cm^2, pyramid B has a surface area of 900 cm^2, and pyramid B has a volume of 810 cm^3, find the volume of pyramid A.

____240 cm^3_______

The ratio of the surface areas $= \frac{400}{900} = \frac{4}{9}$. *The scale factor is therefore* $\sqrt{\frac{4}{9}} = \frac{2}{3}$. *The ratio of the volumes is therefore* $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$. *We can then set up the following ratio to find the volume of pyramid A:*

If we let x represent the volume of pyramid A, then

$$\frac{8}{27} = \frac{x}{810}$$

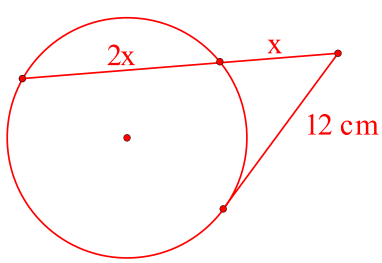
$$27x = 6480$$

$$x = 240$$

The volume of pyramid A is 240 cm^3.

13. A secant and a tangent are drawn to a circle from an external point. The tangent measures 12 cm, and the internal and external segments of the secant have a ratio of 2:1, respectively. Find the length of the secant.

_____$12\sqrt{3}$ cm_____



A diagram is shown at the left. Since the internal and external segments of the secant have a ratio of 2:1, they are labeled as 2x and 1x, respectively.

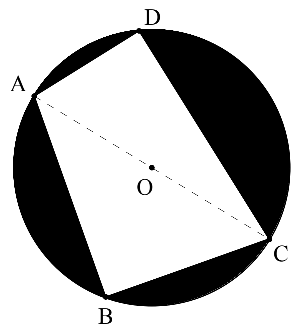
We use the following theorem: If a secant and a tangent intersect in the exterior of a circle, then the product of the lengths of the secant segment and its external segment is equal to the square of the length of the tangent segment.

We can therefore say that $3x(x) = 12^2$. *Simplifying further,*

$$3x^2 = 144$$
$$x^2 = 48$$
$$x = \sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$$

The length of the secant is 3x, so the length of the secant is $12\sqrt{3}$ *cm.*

14. Quadrilateral $ABCD$ is inscribed in circle O of radius 5 cm, with diagonal $\overline{AC}$ passing through the center of the circle. If the length of $\overline{AD}$ is 4 cm and the length of $\overline{AB}$ is 8 cm, find the area of the shaded region.



___$\left(25\pi - 24 - 4\sqrt{21}\right)\text{cm}^2$_____

(These terms can of course be placed in any correct orderl)

ΔADC and ΔABC are right triangles, since they are each inscribed in a semicircle. (To prove this fact, look at angles D and B. Both are inscribed angles and intercept a semicircle which has an arc measure of 180 degrees. Since the measure of an inscribed angle is equal to half the measure of its intercepted arc, then angle D and angle B each measure 90 degrees.)

Since the radius of the circle is 5 cm, the length of $\overline{AC}$ (the hypotenuse of each right triangle) is 10 cm.

Using the Pythagorean Theorem on ΔADC, we can say that

$$4^2 + (DC)^2 = 10^2$$
$$16 + (DC)^2 = 100$$
$$(DC)^2 = 84$$
$$DC = \sqrt{84} = \sqrt{4}\sqrt{21} = 2\sqrt{21}$$

We can then find the area of ΔADC: $\quad Area = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{21}\right)(4) = 4\sqrt{21}\ \text{cm}^2$

To find the missing side $\overline{BC}$ of ΔABC, we notice that the sides represent a Pythagorean triple. We know that one of the legs is 8 cm and the hypotenuse is 10 cm. Since we know that the sides 3:4:5 form a right triangle, then the sides 6:8:10 also represent a right triangle (the side lengths are doubled). We can therefore say that the length of $\overline{BC}$ is 6 cm. (This can also be obtained by using the Pythagorean theorem on the sides of triangle ΔABC.)

We can then find the area of ΔABC: $\quad Area = \frac{1}{2}bh = \frac{1}{2}(6)(8) = 24\ \text{cm}^2$

The area of quadrilateral ABCD is equal to the sums of the areas of ΔADC and ΔADC, so the area of quadrilateral ABCD $= \left(24 + 4\sqrt{21}\right)\ \text{cm}^2$.

The shaded area is the area of the circle minus the area of the quadrilateral. The area of the circle is $\pi r^2 = \pi(5)^2 = 25\pi$. Therefore, the area of the shaded region is

$\left[25\pi - \left(24 + 4\sqrt{21}\right)\right]\ \text{cm}^2$, which is equivalent to $\left(25\pi - 24 - 4\sqrt{21}\right)\text{cm}^2$

15. Name each regular polygon that can tessellate the Euclidean plane.

______equilateral triangle, square, regular hexagon.______

Students may say 'regular triangle' or 'regular quadrilateral.' They should NOT say parallelogram or rectangle or any other polygon that is not specifically regular.

The measure of an interior angle of a regular polygon can be determined by the following fomula:

Measure of one interior angle of a regular polygon $= \frac{180(n-2)}{n}$. *Let us compute the measure of the interior angle for the following regular polygons:*

Equilateral triangle: $= \frac{180(3-2)}{3} = 60^\circ$. *Since 60 divides six times into 360 degrees(a complete rotation), this means that six equilateral triangles can tessellate together at any given vertex. This can continue on and on as they tessellate the entire Euclidean plane.*

Square: $= \frac{180(4-2)}{4} = 90^\circ$. *Since 90 divides four times into 360 degrees (a complete rotation), this means that four squares can tessellate together at any given vertex. This can continue on and on as they tessellate the entire Euclidean plane.*

Regular Pentagon: $= \frac{180(5-2)}{5} = 108^\circ$. *Since 108 does not divide evenly into 360 degrees, so this figure cannot tessellate the entire Euclidean plane.*

Regular Hexagon: $= \frac{180(6-2)}{6} = 120^\circ$. *Since 120 divides three times into 360 degrees (a complete rotation), this means that three regular hexagons can tessellate together at any given vertex. This can continue on and on as they tessellate the entire Euclidean plane.*

If one looks carefully for a pattern they will see that we are done, without needing to do any further computation. Notice that the measure of each interior angle is getting larger as we increase the number of sides. The only number larger then 120 that divides evenly into 360 is 180 – and it is not possible to have a 180 degree angle within a regular polygon.

Therefore, the regular polygons that tessellate the Euclidean plane are the equilateral triangle (or regular triangle), the square (or regular quadrilateral), and the regular hexagon.

16. The perimeter of a square is equal to the perimeter of a circle. Find the ratio of the square's area to the circle's area.

____$\pi:4$ or $\dfrac{\pi}{4}$__

(Do NOT accept $\dfrac{4}{\pi}$ or $4:\pi$)

Let x represent the side of the square. Then

$P = 4x$, *where P is the perimeter of the square,*

$A_1 = x^2$, *where* A_1 *is the area of the square.*

Let r represent the radius of the circle. Then

$P = 2\pi r$, *where P is the perimeter (circumference) of the circle,*

$A_2 = \pi r^2$, *where* A_2 *is the area of the circle.*

We want to represent both areas in terms of P (since the perimeters are the same for both the circle and the square).

For the square, since $P = 4x$, *we can say that* $x = \dfrac{P}{4}$, *so* $A_1 = x^2 = \left(\dfrac{P}{4}\right)^2 = \dfrac{P^2}{16}$.

For the circle, since $P = 2\pi r$, *we can say that* $r = \dfrac{P}{2\pi}$, *so* $A_2 = \pi\left(\dfrac{P}{2\pi}\right)^2 = \dfrac{\pi P^2}{4\pi^2} = \dfrac{P^2}{4\pi}$.

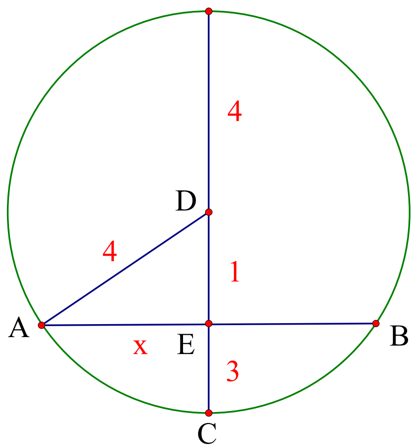
The ratio of the square's area to the circle's area is shown below:

$$\frac{A_1}{A_2} = \frac{\frac{P^2}{16}}{\frac{P^2}{4\pi}} = \frac{P^2}{16} \cdot \frac{4\pi}{P^2} = \frac{\pi}{4}$$

The ratio of the square's area to the circle's area is $\dfrac{\pi}{4}$ *(or equivalently,* $\pi:4$ *or "*π *to 4.")*

17. A spherical balloon contains water that is 3 inches deep at the deepest point. If the balloon has a diameter of 8 inches, what is the area of the top surface of the water?

______15π in^2_____



The diagram at the left represents a vertical cross section of the sphere. Since the diameter of the sphere is 8 cm, the radius of the sphere is 4 cm. Since the length of $\overline{DC}$ (a radius) is 4 and the water is 3 inches deep, we conclude that the length of $\overline{DE}$ is 1.

Using the Pythagorean Theorem on ΔADE,

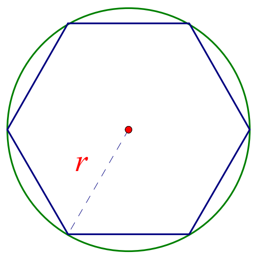
$$x^2 + 1^2 = 4^2$$
$$x^2 + 1 = 16$$
$$x^2 = 15$$
$$x = \sqrt{15}$$

Since x represents the radius of the top surface of the water, the area of the top surface of the water is found below:

$$A = \pi x^2 = \pi\left(\sqrt{15}\right)^2 = 15\pi \text{ in}^2.$$

So the area of the top surface of the water is 15π in^2.

18. A circle circumscribes a regular hexagon which has area $30\sqrt{3}$ cm^2. How long is the circular arc between two adjacent vertices of the hexagon?

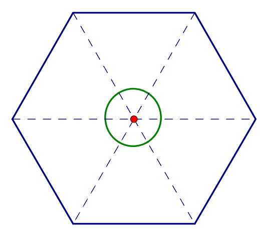


$\frac{2\pi}{3}$

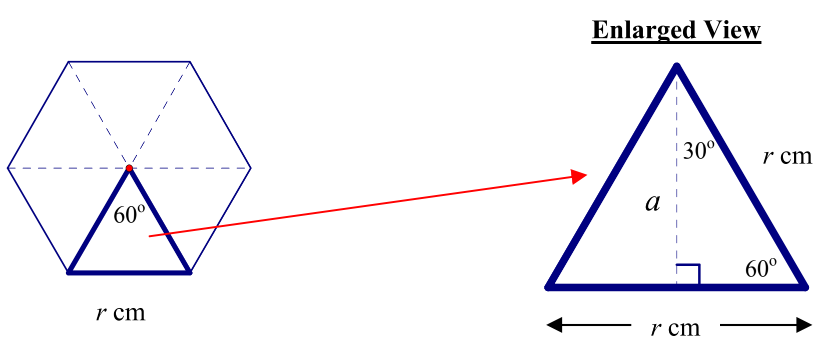
(or some other equivalent simplified answer.)

The diagram above shows a circle of radius r which circumscribes a regular hexagon. If we find the length of the radius, we can then compute the length of the circular arc between two adjacent vertices of the regular hexagon.

Let us first look at the hexagon more closely:



Let us examine the properties of one of the six triangles above. Since a regular hexagon has 6 sides, the measure of one of its central angles is $\frac{360^\circ}{6} = 60^\circ$. *Since the radii are all congruent, the six triangles are each isosceles triangles and in fact are equilateral, since each central angle measures 60 degrees. Therefore, each side of the hexagon also has length r. If an apothem is drawn, we can see that it forms a* 30^o*-*60^o*-*90^o *triangle, as shown in the enlarged view of the triangle below.*



The formula for the area of a regular polygon is $A = \frac{1}{2}aP$*, where a is the apothem (shown in the previous diagram) and P is the perimeter of the regular polygon.*

$P = 6r$*, since all the sides of the regular hexagon are congruent and are equal to the radius of the circle.*

We need to write the apothem in terms of r.

In the 30^o*-*60^o*-*90^o *triangle on the previous page, the side opposite the 30 degree angle is* $\frac{r}{2}$*. Therefore, the apothem a is* $\frac{r}{2}\sqrt{3}$*. (This is a shortcut for* 30^o*-*60^o*-*90^o *triangles but can also be obtained using the Pythagorean Theorem.)*

Therefore, the area of the regular polygon can be represented as follows:

$$A = \frac{1}{2}aP = \frac{1}{2}\left(\frac{r}{2}\sqrt{3}\right)(6r) = \frac{6r^2\sqrt{3}}{4} = \frac{3r^2\sqrt{3}}{2}$$

It was given in the initial information for the problem that the $A = 30\sqrt{3}\ cm^2$*, so*

$$\frac{3r^2\sqrt{3}}{2} = 30\sqrt{3}.$$ *Solving for r,*

$$3r^2\sqrt{3} = 60\sqrt{3}$$
$$r^2 = 20$$
$$r = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}\ cm$$

We now need to find the length of the circular arc between adjacent vertices. We can set up the proportion:

$$\frac{60}{360} = \frac{x}{C}$$ *, where x represents the arclength and C represents the circumference.*

Since $C = 2\pi r$*, we know that* $C = 2\pi\left(2\sqrt{5}\right) = 4\pi\sqrt{5}$*.*

So $\frac{60}{360} = \frac{x}{4\pi\sqrt{5}}$*, or equivalently* $\frac{1}{6} = \frac{x}{4\pi\sqrt{5}}$*.*

$$6x = 4\pi\sqrt{5}$$

$$x = \frac{4\pi\sqrt{5}}{6} = \frac{2\pi\sqrt{5}}{3}\ cm$$

Therefore, the length of the circular arc between adjacent vertices is $\frac{2\pi\sqrt{5}}{3}\ cm$*.*

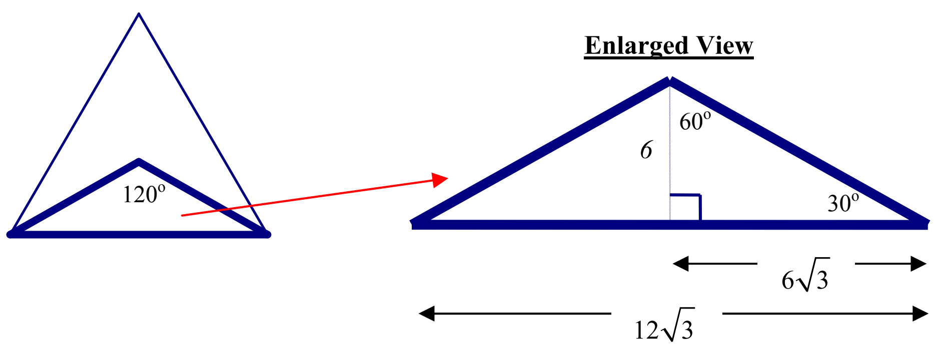
19. A regular right triangular pyramid has altitude 8 cm and slant height 10 cm. Find the volume of the pyramid.

____ $288\sqrt{3}\ \text{cm}^3$ ____

A right triangle can be drawn inside the pyramid with the slant height as the hypotenuse, the altitude as one of the legs, and the apothem of the base as the other leg. Using Pythagorean triples (or the Pythagorean Theorem), the apothem of the base can be found to be 6 cm.

Let us next look more closely at the base, which is an equilateral triangle:

In a 30^o-60^o-90^o triangle, the length of the hypotenuse is twice the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg. Since the shorter leg is 6 cm, the longer leg is $6\sqrt{3}\ cm$ as shown in the diagram below. Since the longer leg is $6\sqrt{3}\ cm$, each side of the equilateral triangle is $12\sqrt{3}\ cm$.



Since each side of the equilateral triangle is $12\sqrt{3}\ cm$, the perimeter of the equilateral triangle is $3\left(12\sqrt{3}\right)=36\sqrt{3}\ cm$.

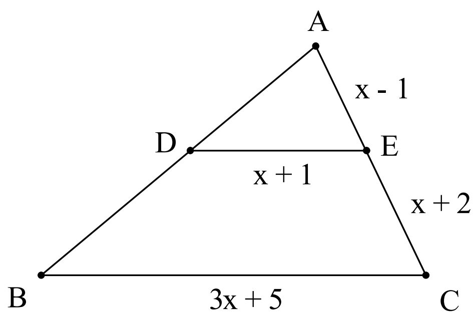
The area of the base of the pyramid (the equilateral triangle)) is
$B=\frac{1}{2}aP=\frac{1}{2}(6)\left(36\sqrt{3}\right)=108\sqrt{3}\ cm^2$.

The formula for the volume of a pyramid is $V=\frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid.

So $V=\frac{1}{3}Bh=\frac{1}{3}\left(108\sqrt{3}\right)(8)=288\sqrt{3}\ cm^3$.

20. In ΔABC below, $\overline{DE} \parallel \overline{BC}$, $AE = x-1$, $EC = x+2$, $DE = x+1$, and $BC = 3x+5$. Find x.

________3___________



Since $\overline{DE} \parallel \overline{BC}$, $\Delta ADE \sim \Delta ABC$ by Angle-Angle Similarity. Therefore the corresponding sides of the two triangles are proportional, so we can say that:

$$\frac{AE}{AC} = \frac{DE}{BC}$$

Notice that $AC = (x-1)+(x+2) = 2x-1$

So $\dfrac{x-1}{2x+1} = \dfrac{x+1}{3x+5}$.

Cross-multiplying, we obtain:

$(x-1)(3x+5) = (2x+1)(x+1)$

$3x^2 + 2x - 5 = 2x^2 + 3x + 1$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

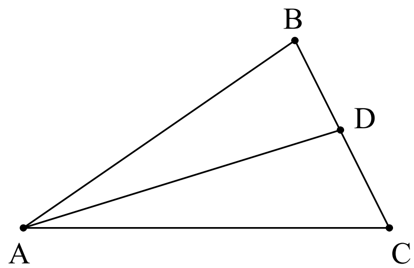
$x = 3$

($x = -2$ *is an extraneous solution, since side lengths of the triangles need to be positive.)*

Therefore $x = 3$.

21. $\overline{AD}$ is an angle bisector of ΔABC. If the length of $\overline{AB}$ is 10 cm, the length of $\overline{BD}$ is 4 cm, and the length of $\overline{DC}$ is 6 cm, find the length of $\overline{AC}$.

_____15 cm________



We will use the following theorem: If a ray bisects the angle of a triangle, then it divides the opposite side into segments proportional to the other two sides of the triangle.

We can then say that $\frac{BD}{DC} = \frac{AB}{AC}$.

Substituting the known values, $\frac{4}{6} = \frac{10}{AC}$

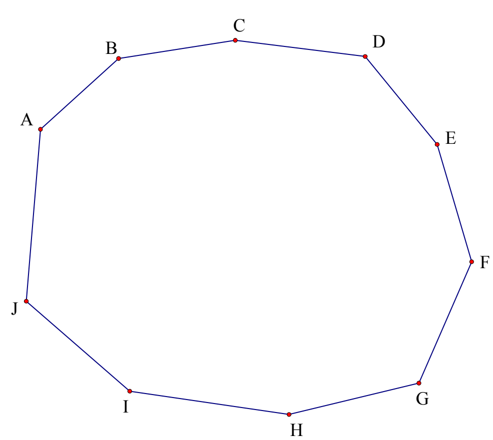
$$4AC = 60$$
$$AC = 15$$

Therefore, the length of $\overline{AC}$ *is* 15 cm.

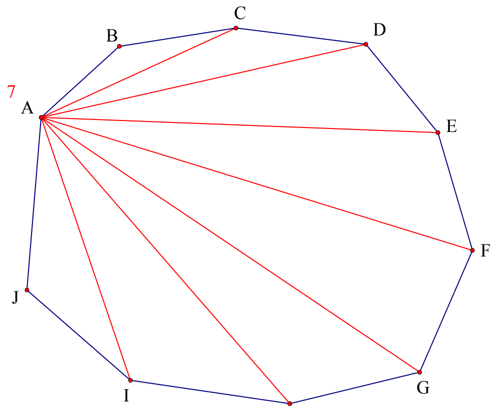
22. Find the number of diagonals in a convex decagon.

_______35_________

A shortcut formula can be found below, but the standard method will be described first. We begin by drawing a decagon and labeling the vertices A through J, as shown below.

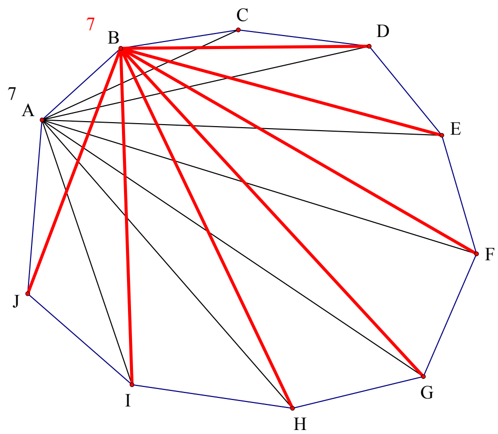


We then start at point A and draw as many diagonals as we can. We find that we can draw 7 diagonals. (Notice that we have written the number 7 next to vertex A, to be able to easily recount the diagonals later.)



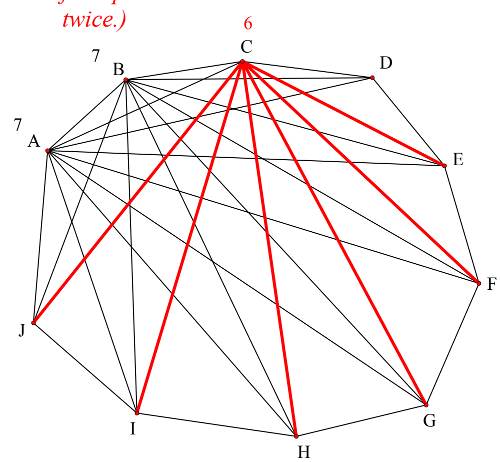
We then repeat the same process from point B, then point C, etc. The diagrams for each successive step are shown below, with new diagonals drawn in red.

New diagonals from point B:



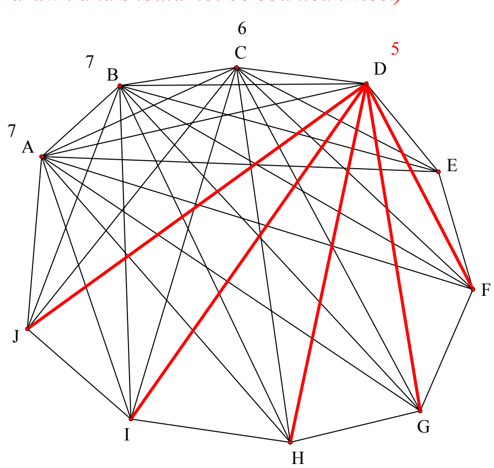
New diagonals from point C:

(Notice that $\overline{AC}$ has already been drawn from point A and should not be counted twice.)

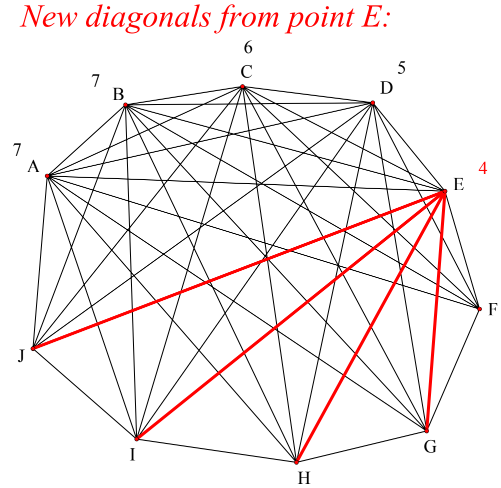


New diagonals from point D:

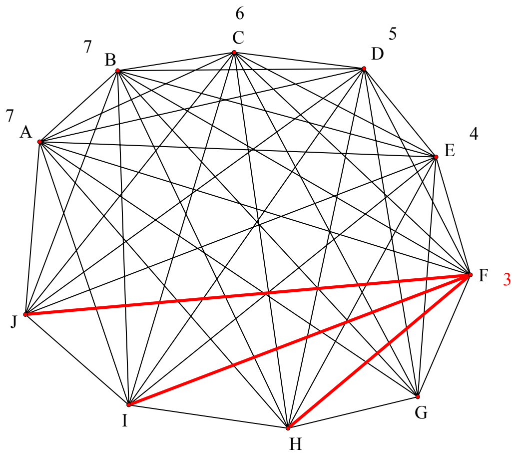
(Notice that $\overline{AD}$ and $\overline{BD}$ have already been drawn and should not be counted twice.)



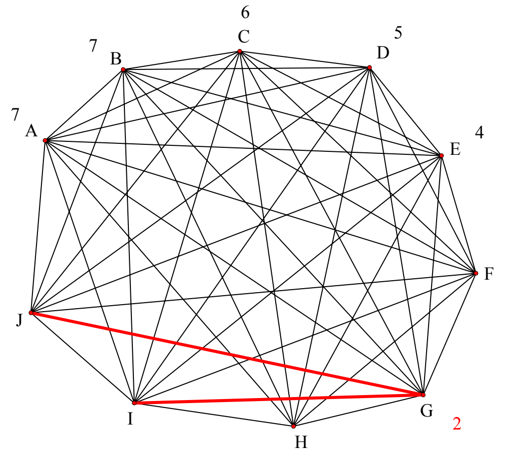
New diagonals from point E:



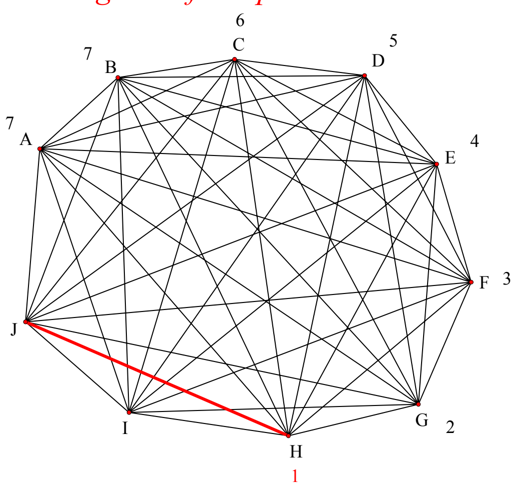
New diagonals from point F:



New diagonals from point G:



New diagonals from point H:



With the method we have used, no new diagonals can be drawn from vertices I or J.
Therefore, the total number of diagonals is $7+7+6+5+4+3+2+1=35$.
So there are 35 diagonals in a convex decagon.

Note:
The shortcut formula for the number of diagonals in a convex polygon with n sides is:

$$\frac{n(n-3)}{2}$$

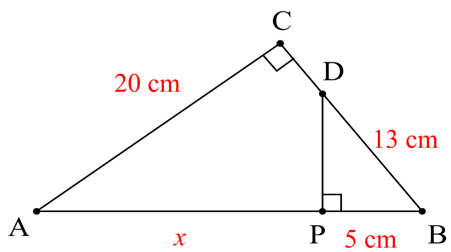
Using this formula and the fact that $n=10$, *the number of diagonals is*

$$\frac{10(10-3)}{2}=\frac{10(7)}{2}=35.$$

23. In the figure below, the measure of $\overline{AC}$ is 20 cm, the measure of $\overline{DB}$ is 13 cm, and the measure of $\overline{PB}$ is 5 cm. Find the measure of $\overline{AP}$.

___ $\frac{50}{3}$ cm ___

(The answer $16\frac{2}{3}$ cm is also fine.)



We can first find the length of $\overline{DP}$ *using either Pythagorean triples or the Pythagorean Theorem, and we find that* $DP = 12$.

Using right triangle trigonometry on ΔDPB, *we find that* $\sin(B) = \frac{12}{13}$.

If we let x represent the length of $\overline{AP}$ *and use right triangle trigonometry on* ΔACB, *we find that* $\sin(B) = \frac{20}{x+5}$.

Notice that we have found two different representations for $\sin(B)$. *Setting them equal to each other, we obtain the following:*

$$\frac{12}{13} = \frac{20}{x+5}$$

$$12x + 60 = 260$$

$$12x = 200$$

$$x = \frac{200}{12} = \frac{50}{3}$$

Therefore, the measure of $\overline{AP}$ *is* $\frac{50}{3}$ *cm*.

24. The sum of the measures of all but one of the interior angles of a convex polygon is 1030°. Find the measure of the remaining angle.

_____ 50° __________

50 (with no degree symbol) is also fine.

The formula for the sum of the interior angles of an n-sided convex polygon is:

Sum of the interior angles $=(n-2)\cdot 180$.

Therefore, the sum of the interior angles of a convex polygon is always a multiple of 180. If we list multiples of 180, we obtain the following:
180, 360, 540, 720, 900, 1080, 1260, …

All the multiples of 180 up to and including 900 do not work in this problem, as the sum of the angles is too small. If the sum of the angles is 1080 degrees, then the remaining angle is 50 degrees.

The next question is whether or not the sum of the angles could be 1260 degrees. The answer is no, since the remaining angle would be greater than 180 degrees, which would make the polygon concave rather than convex.

Therefore, the measure of the remaining angle is 50°.

25. A rectangular box has a length of 6 cm, a depth of 3 cm, and a height of 2 cm. What is the length of the longest pencil that can fit inside the box?

_____ 7 cm _________

The length d of a diagonal of a rectangular prism with dimensions x, y, and z can be represented by the following formula:

$$d=\sqrt{x^2+y^2+z^2}$$

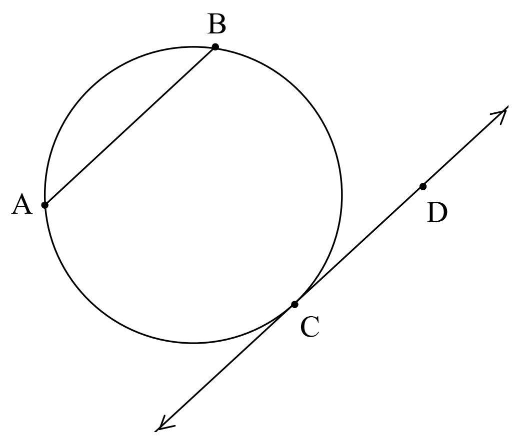
$$d=\sqrt{x^2+y^2+z^2}=\sqrt{6^2+3^2+2^2}=\sqrt{36+9+4}=\sqrt{49}=7$$

Therefore, the longest pencil that can fit inside the box is 7 cm.

Note: Rather than the shortcut formula, the Pythagorean Theorem can be used twice. First, the Pythagorean Theorem can be used along the 6 cm x 3cm rectangular base to obtain a hypotenuse of $\sqrt{45}=3\sqrt{5}$ *cm. Then this new measurement of* $3\sqrt{5}$ *cm is used as a leg along with the height of 2 cm as the other leg, to find the hypotenuse which is the diagonal of the box. This of course still yields a result of 7 cm.*

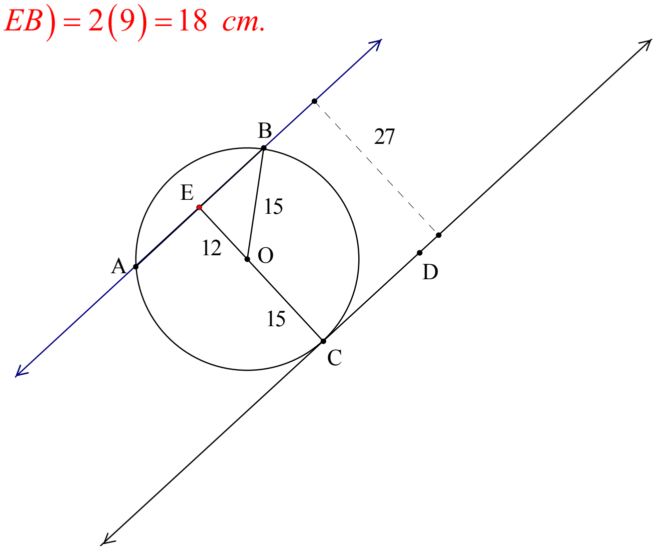
26. $\overleftrightarrow{CD}$ is tangent to the circle below at point C, and chord $\overline{AB}$ is parallel to tangent line $\overleftrightarrow{CD}$. If the radius of the circle is 15 cm, and the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is 27 cm, find the length of chord $\overline{AB}$.

_____18 cm________



In the diagram below, we have labeled the center of the circle as O, and the midpoint of $\overline{AB}$ as E. Since the distance between the parallel lines is 27 and the radius of the circle is 15 cm, we can conclude that $\overline{EO}$ has length $27-15=12$ cm. We draw in radius $\overline{OB}$ which is the hypotenuse of right ΔEOB. Using Pythagorean Triples or the Pythagorean Theorem, we find that the length of $\overline{EB}$ is 9 cm. Since E is the midpoint of $\overline{AB}$,

$$AB=2(EB)=2(9)=18 \text{ cm.}$$



27. Find the equation of the line tangent to the circle $(x-2)^2+(y+1)^2=25$ at the point $(5,3)$, and write the equation in slope-intercept form.

__ $y=-\frac{3}{4}x+\frac{27}{4}$ __

(or some other equivalent simplified answer, like $y=-0.75x+6.75$, or $y=-\frac{3}{4}x+6\frac{3}{4}$)

The circle $(x-2)^2+(y+1)^2=25$ *has center* $(2,-1)$. *The slope of the line containing the radius from* $(2,-1)$ *to* $(5,3)$ *is:*

$$slope=\frac{3-(-1)}{5-2}=\frac{4}{3}$$

The tangent line has at the point $(5,3)$ *has slope* $-\frac{3}{4}$, *since the tangent line is perpendicular to the radius. (Perpendicular lines have slopes that are negative reciprocals of each other; i.e. their slopes have a product of* -1.*)*

The tangent line has slope $-\frac{3}{4}$ *and passes through the point* $(5,3)$, *so we write an equation of the tangent line in point-slope form:*

$$y-3=-\frac{3}{4}(x-5)$$

We solve for y to write the equation in slope-intercept form:

$$y-3=-\frac{3}{4}x+\frac{15}{4}$$

$$y=-\frac{3}{4}x+\frac{15}{4}+3$$

$$y=-\frac{3}{4}x+\frac{15}{4}+\frac{12}{4}$$

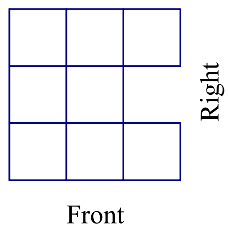
$$y=-\frac{3}{4}x+\frac{27}{4}$$

Therefore, the equation of the tangent line to the circle at the point $(5,3)$ *is* $y=-\frac{3}{4}x+\frac{27}{4}$.

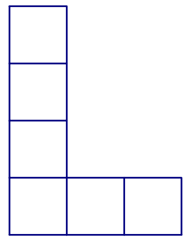
28. A building is made from stacked cubes, and views from five different perspectives are illustrated below. How many cubes are needed to construct this building?

______14____________

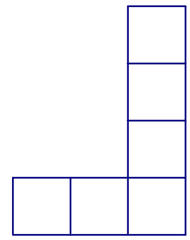
Top View:



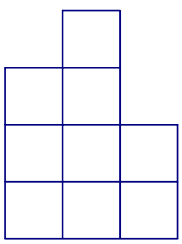
Front View:



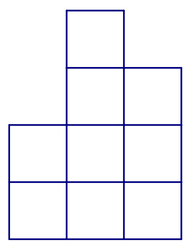
Back View:



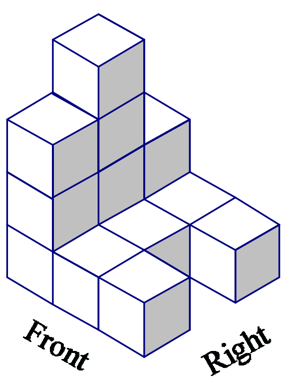
Right View:



Left View:



There is only one building made of cubes which fits all of the perspectives listed above. The building is shown below in both a three dimensional perspective, as well as the perspective from the top with the number of cubes in each stack.



Top View

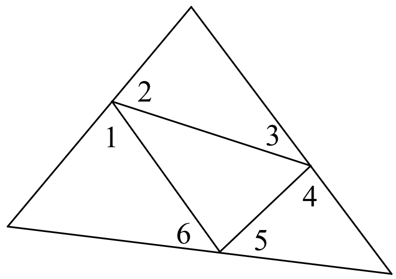
2	1	1
4	1	
3	1	1

Right

Front

The building is made up of 14 cubes.

29. Find the sum of the degree measures of the numbered angles in the diagram below.



______360°________

360 (with no degree symbol) is also fine.

Let x, y, and z represent the measures of the angles of the inner triangle as shown below.

$x + m\measuredangle 1 + m\measuredangle 2 = 180$, *so* $x = 180 - m\measuredangle 1 - m\measuredangle 2$.
Similarly,
$y = 180 - m\measuredangle 3 - m\measuredangle 4$, *and* $z = 180 - m\measuredangle 5 - m\measuredangle 6$.

Since x, y, and z represent the measures of the angles of a triangle, their sum is 180 degrees. Therefore,

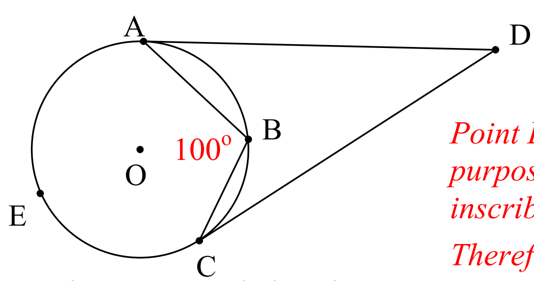
$$x + y = z = 180$$
$$(180 - m\measuredangle 1 - m\measuredangle 2) + (180 - m3 - m\measuredangle 4) + (180 - m\measuredangle 5 - m\measuredangle 6) = 180$$
$$540 - m\measuredangle 1 - m\measuredangle 2 - m\measuredangle 3 - m\measuredangle 4 - m\measuredangle 5 - m\measuredangle 6 = 180$$
$$360 = m\measuredangle 1 + m\measuredangle 2 + m\measuredangle 3 + m\measuredangle 4 + m\measuredangle 5 + m\measuredangle 6$$

Therefore, the sum of the numbered angles in the figure is 360 degrees.

30. $\overline{AD}$ and $\overline{DC}$ are tangents to circle *O*, and $\overline{AB}$ and $\overline{BC}$ are chords. If the measure of $\angle ABC$ is 100°, find the measure of $\angle ADC$.

______20°______

20° (with no degree symbol) is also fine



Point E has been added to the diagram for naming purposes. $\measuredangle ABC$ *is an inscribed angle, and an inscribed angle measures half of its intercepted arc. Therefore, the measure of* $\overset{\frown}{AEC}$ *is 200 degrees. Since the entire circle has degree measure 360 degrees, the measure of* $\overset{\frown}{ABC} = 360 - 200 = 160$. *When two tangents intersect outside a circle, the measure of the angle formed by the tangents is half the difference of the measures of the intercepted arcs. Therefore, the measure of* $\measuredangle ADC = \dfrac{200-160}{2} = \dfrac{40}{2} = 20$ *degrees.*

31. If the diameter of a spherical balloon is 2 meters, how much air must be blown into the balloon to quadruple its surface area?

____ $\frac{28\pi}{3}$ m^3 ______

The initial radius of the sphere is 1 meter, so the initial volume V_1 of the sphere is found below:

$$V_1 = \frac{4\pi r^3}{3} = \frac{4\pi(1)^3}{3} = \frac{4\pi}{3}\,m^3$$

We want to quadruple the surface area, which means that the surface area ratio is 4:1. Therefore, the scale factor between the radii of the two spheres is 2:1. So we need to double the radius of the sphere.

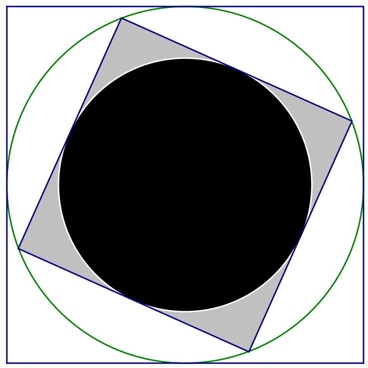
If we double the radius of the sphere, then the new radius is 2 meters, so the new volume V_2 is:

$$V_2 = \frac{4\pi r^3}{3} = \frac{4\pi(2)^3}{3} = \frac{4\pi(8)}{3} = \frac{32\pi}{3}\,m^3$$

Therefore, the amount of air which must be blown into the balloon is:

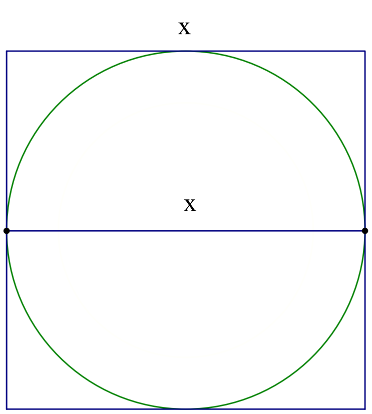
$$V_2 - V_1 = \left(\frac{32\pi}{3} - \frac{4\pi}{3}\right) = \frac{28\pi}{3}\,m^3$$

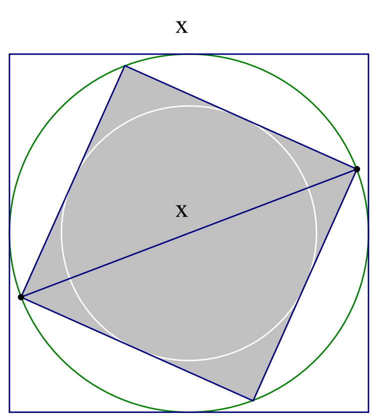
32. The dartboard below is comprised entirely of squares and circles. (A square is circumscribed about a circle which is circumscribed about a square which is circumscribed about a circle.) If a dart is thrown at the dartboard, what is the probability that it hits the gray shaded region -- within the smallest square but outside the smallest circle? (Assume that all darts thrown are randomly distributed within the outer square.)

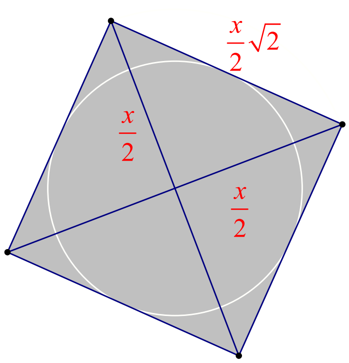


$\frac{1}{2}-\frac{\pi}{8}$, or $\frac{4-\pi}{8}$

If we label the side of the largest square as x, then the diameter of the largest circle is also x, as shown in the figure at the left below. This means that the diagonal of the inner square is also x, as shown in the figure at the right below.

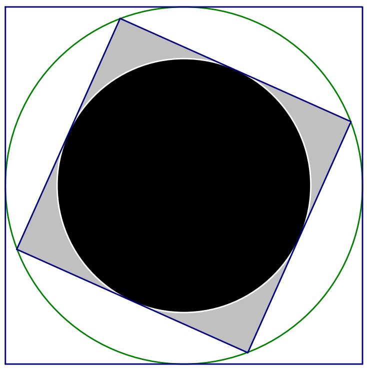






If we draw the other diagonal of the inner shaded square, we can see that the diagonals divide the square into four 45^o-45^o-90^o triangles. Each leg is $\frac{x}{2}$*, so the hypotenuse – which is the side of the shaded square, measures* $\frac{x}{2}\sqrt{2}$. *Therefore, the area of the shaded square is* $\left(\frac{x}{2}\sqrt{2}\right)^2=\frac{x^2}{4}(2)=\frac{x^2}{2}$.

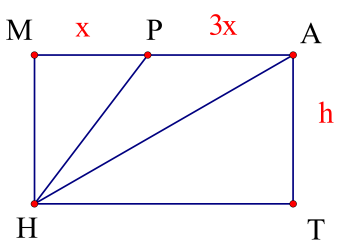
The diameter of the inner circle is the same as the side of the square, so the diameter of the inner circle is $\frac{x}{2}\sqrt{2}$. *Therefore, the radius of the inner square is* $\frac{1}{2}\cdot\frac{x}{2}\sqrt{2}=\frac{x}{4}\sqrt{2}$. *So the area of the inner circle is* $\pi\left(\frac{x}{4}\sqrt{2}\right)^2=\pi\frac{x^2}{16}(2)=\frac{\pi x^2}{8}$.



The area of the gray shaded region within the smallest square but outside the smallest circle is therefore $\frac{x^2}{2}-\frac{\pi x^2}{8}$. *So the probability that the dart would land within this gray shaded region on the dartboard is* $\frac{\frac{x^2}{2}-\frac{\pi x^2}{8}}{x^2}=\frac{x^2\left(\frac{1}{2}-\frac{\pi}{8}\right)}{x^2}=\frac{1}{2}-\frac{\pi}{8}$.

Therefore, the desired probability is $\frac{1}{2}-\frac{\pi}{8}$, *which can also be written as* $\frac{4-\pi}{8}$.

33. In rectangle *MATH*, point *P* is on side $\overline{MA}$ such that the ratio of $\overline{MP}$ to $\overline{PA}$ is 1:3. What is the ratio of the area of ΔPAH to the area of rectangle *MATH* ?



$3:8$, or $\frac{3}{8}$, or "3 to 8"

____

Since the ratio MP:PA, we have labeled $\overline{MP}$ *as x and* $\overline{PA}$ *as 3x. The area of triangle PAH is then* $\frac{1}{2}(3xh)=\frac{3xh}{2}$. *The area of rectangle MATH is 4xh. Therefore, the ratio of the area of triangle PAH to the area of rectangle MATH is* $\frac{\frac{3xh}{2}}{4xh}=\frac{3xh}{2}\cdot\frac{1}{4xh}=\frac{3}{8}$.

34. The hypotenuse of a right triangle measures 4 cm and its perimeter is 9 cm. Find the area of the triangle.

$\frac{9}{4}$ cm^2

Other acceptable answers are $2\frac{1}{4}$ cm or 2.25 cm.)

If we let x and y represent the legs of the right triangle, then $x^2 + y^2 = 16$.

Since the perimeter is 9 cm, $x + y + 4 = 9$, *so* $x + y = 5$.

Therefore, $y = 5 - x$.

Substituting this into the original equation,

$$x^2 + (5-x)^2 = 16$$

$$x^2 + 25 - 10x + x^2 = 16$$

$$2x^2 - 10x + 9 = 0$$

We use the quadratic formula to find x, and find that:

$$x = \frac{10 \pm 2\sqrt{7}}{4}.$$

If we look at the solution $x = \frac{10+2\sqrt{7}}{4}$, *then* $y = 5 - x = 5 - \left(\frac{10+2\sqrt{7}}{4}\right) = \frac{10-2\sqrt{7}}{4}$.

Therefore the area of the triangle is as follows:

$$A = \frac{1}{2}xy = \frac{1}{2}\left(\frac{10+2\sqrt{7}}{4}\right)\left(\frac{10-2\sqrt{7}}{4}\right) = \frac{1}{2}\left(\frac{100-28}{16}\right) = \frac{1}{2}\left(\frac{72}{16}\right) = \frac{72}{32} = \frac{9}{4}\ cm^2.$$

35. A spherical glass ornament of radius 10 cm has a cube manufactured inside it. What is the volume of the largest cube that can fit inside the ornament?

$\frac{8000\sqrt{3}}{9}$ cm^3

The diagonal of the cube is the same as the diameter of the sphere, and the diameter of the sphere is 20 cm. If we let x represent the side of the cube, we obtain the following equation:

$$20 = \sqrt{x^2 + x^2 + x^2}$$

$$20 = \sqrt{3x^2}$$

$$: 400 = 3x^2$$

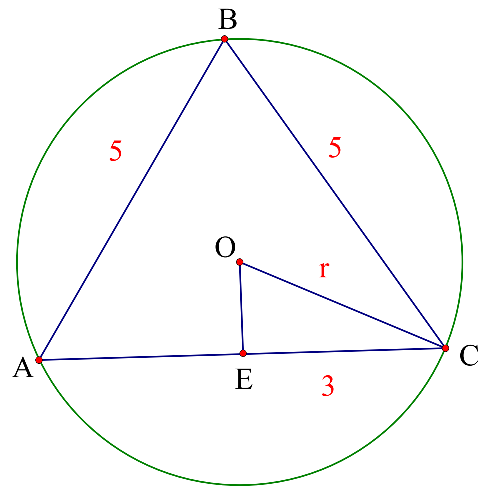
$$x^2 = \frac{400}{3}$$

$$x = \sqrt{\frac{400}{3}} = \frac{20}{\sqrt{3}}$$

Therefore, the volume of the cube is

$$V = \left(\frac{20}{\sqrt{3}}\right)\left(\frac{20}{\sqrt{3}}\right)\left(\frac{20}{\sqrt{3}}\right) = \frac{8000}{3\sqrt{3}} = \frac{8000}{3\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{8000\sqrt{3}}{3(3)} = \frac{8000\sqrt{3}}{9}\ cm^3$$

36. Circle O is circumscribed about ΔABC, where the length of $\overline{AB}$ is 5 cm, the length of $\overline{BC}$ is 5 cm, and the length of $\overline{AC}$ is 6 cm. Find the length of the radius of circle O.



____ $\frac{25}{8}$ cm ________

(Other acceptable answers are $3\frac{1}{8}$ cm or 3.125 cm.)

$\measuredangle B \cong \measuredangle EOC$, since they both equal half the measure of $\overset{\frown}{AC}$.
The area of triangle ABC can be represented by

$K = \frac{1}{2}ac\sin(B) = \frac{1}{2}(5)(5)\sin(B) = \frac{25}{2}\sin(B)$

$\sin(EOC) = \frac{3}{r}$, *so* $\sin(B) = \frac{3}{r}$.

So $K = \frac{75}{2r}$.

But if we draw the altitude from point B in triangle ABC, we find that it is 4 cm (since the other leg is 3 cm and the hypotenuse is 5 cm.

So the area of the triangle is $K = \frac{6(4)}{2} = 12$.

Setting the two area expressions equal to each other, we find that

$$\frac{75}{2r} = 12$$
$$24r = 75$$
$$r = \frac{75}{24} = \frac{25}{8} \text{ cm}$$

(Note: The shortcut formula for this problem is $r = \frac{abc}{4K}$ *.)*