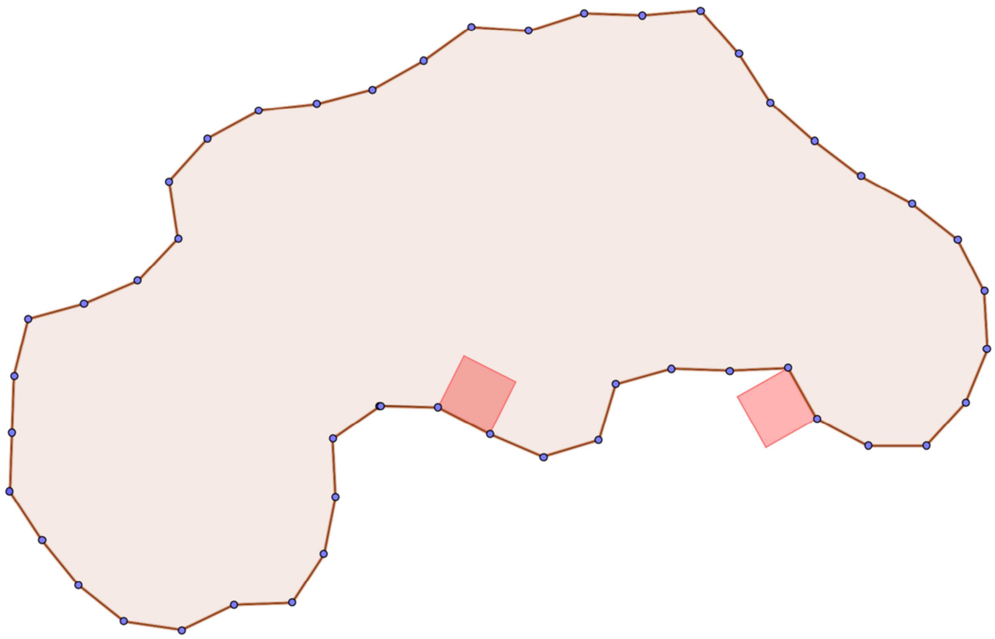
University of Houston – Math Contest
Project Problem, 2012

Simple Equilateral Polygons and Regular Polygonal Wheels

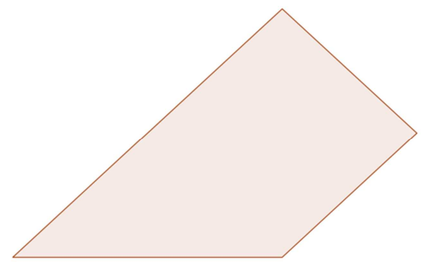


School Name: ______________________________________

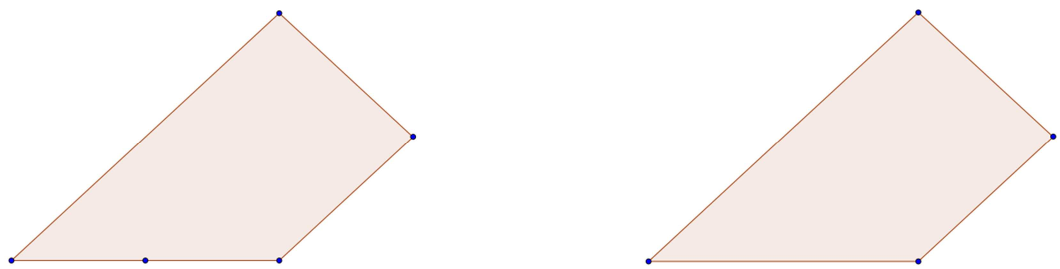
Team Members	

The Project is due at 8:30am (during onsite registration) on the day of the contest. **Each project will be judged for precision, presentation, and creativity.** Place the team solution in a folder with this page as the cover page, a table of contents, and the solutions. Solve as many of the problems as possible.

A polygon is said to be **simple** if its boundary does not cross itself. A polygon is **equilateral** if all of its sides have the same length. The sides of a polygon are line segments joining particular vertices of the polygon, and quite often, it is necessary to specify the vertices of a polygon, even if a figure is given. For example, consider the simple polygon shown below.

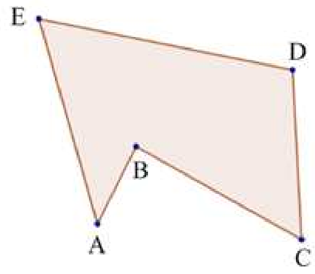


In the absence of additional information, we naturally assume that this simple polygon has 4 vertices and 4 sides. However, it is possible that there are more vertices and sides. Consider the two views below of figures with the same shape, with the vertices showing.



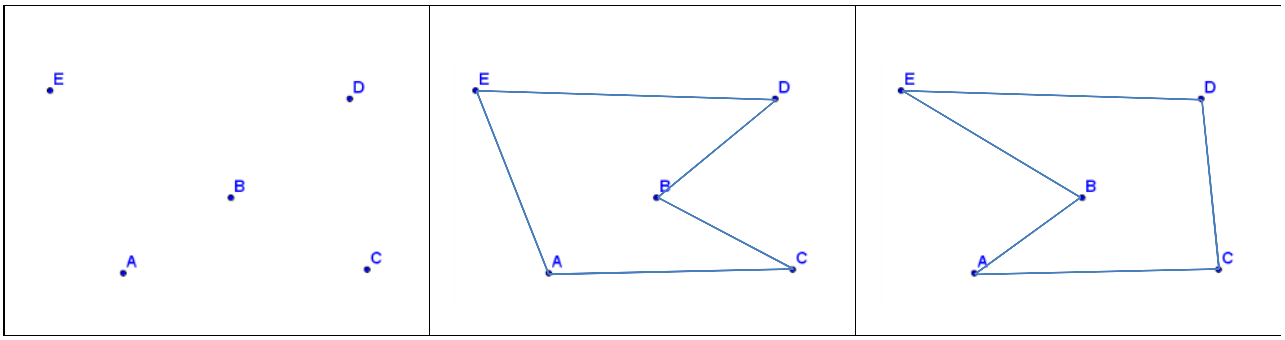
The first of these has an additional vertex, and consequently an additional side (it really is possible for 2 sides to be adjacent and collinear). For this reason, whenever referring to a simple polygon, we will either clearly plot the vertices, or list them.

We assume the reader understands the difference between the **interior** and the **exterior** of a simple polygon, as well as the idea of traversing the boundary of a simple polygon in a **counter-clockwise orientation**. For example, one counter-clockwise orientation of the polygon below could be described as a trip from A to B to C to D to E to A.



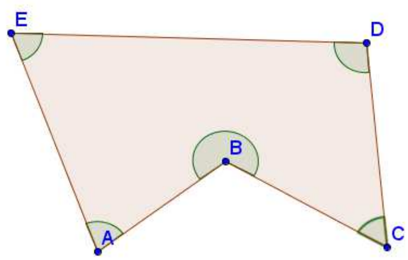
The idea of referring to counter-clockwise orientation allows us to describe the figure above as polygon ABCDE or polygon BCDEA or polygon CDEAB or polygon DEABC or polygon EABCD, with each of these descriptions giving the same simple polygon.

Note that if the vertices of a simple polygon are listed in counter-clockwise order, then the sides of the simple polygon are known. This is not true if the vertices are just given (in arbitrary order). For example, consider the points shown below, followed by two different polygons that have these points as vertices.



The figure in the middle shows polygon ACBDE, and the figure on the right shows polygon ACDEB.

The **interior angles** of a simple polygon can be described once the vertices are given in a counter-clockwise orientation. We illustrate this with the polygon ABCDE shown below with its interior angles shaded.

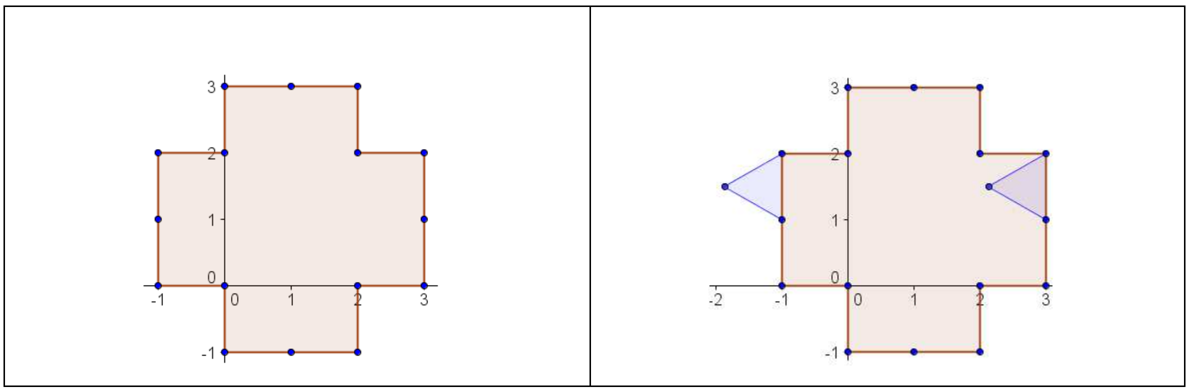


Note that the interior angle at vertex A is determined by rotating counter-clockwise about the point A, from the point B to the ray $\overrightarrow{AE}$. Similarly, the interior angle at vertex B is determined by rotating counter-clockwise about the point B, from the point C to the ray $\overrightarrow{BA}$. In general, if $n \geq 3$ is a natural number and $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, …, $P_n = (x_n, y_n)$ are points in the plane so that polygon $P_1 P_2 \cdots P_n$ is a (counter-clockwise description of a) simple polygon, then we can describe the interior angles of the polygon as follows. First, denote $P_0 = P_n$ and $P_{n+1} = P_1$. Then, for each $i = 1, \ldots, n$, the interior angle at P_i is determined by rotating counter-clockwise about the point P_i, from the point P_{i+1} to the ray $\overrightarrow{P_i P_{i-1}}$.

A **regular polygon** is a simple equilateral polygon in which all of the interior angles have the same measure. Note that not all definitions of "regular polygon" require the polygon to be simple, but we make this assumption here.

Problems – Rolling regular polygonal wheels on equilateral simple polygonal tracks.

1. Consider the simple polygon shown on the left below, and the same polygon shown on the right along with two equilateral triangles.

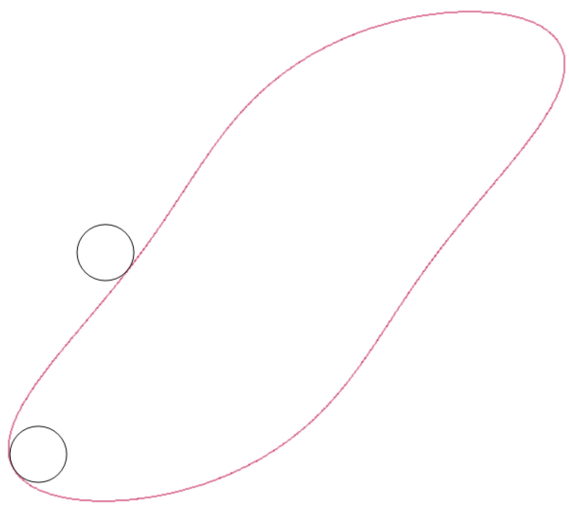


Imagine that the equilateral triangles are wheels and the simple polygon with 16 vertices is a track. The equilateral triangle in the interior of the track "rolls" around the inside of the track. As it rolls, it stays on the inside of the track, and always has one of its vertices attached to a vertex of the track so that it can rotate clockwise about this vertex, until another vertex of the triangle can attach to another vertex of the track. The equilateral triangle in the exterior of the track "rolls" around the outside of the track. As it rolls, it stays on the outside of the track, and it always has one of its vertices attached to a vertex of the track so that it can rotate counter-clockwise about this vertex, until another vertex of the triangle can attach to another vertex of the track.

   a. Give the total measure of the angle of clockwise rotation that the interior wheel will make as it rolls one time around the interior of the track.
   b. Give the total measure of the angle of counter-clockwise rotation that the exterior wheel will make as it rolls one time around the exterior of the track.
   c. Plot the track along with the curves that the vertices of the wheels will travel on as the wheels roll around the inside and outside of the track.
   d. Give the total distance traveled by each of the vertices of the wheels as they roll once around the track (both for the wheel on the inside and the one on the outside).

2. The **diameter** of a polygon is the largest possible distance between any two points of the polygon.
    a. Give the diameter of the polygonal track in problem 1.
    b. Give a formula for the diameter of a regular polygon in terms of the number of sides and the common lengths of the sides.
    c. Prove the formula in part b.
3. Suppose, $n \geq 3$ is a natural number and $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, …, $P_n = (x_n, y_n)$ are points in the plane so that polygon $P_1P_2 \cdots P_n$ is a (counter-clockwise description of a) simple polygon with n sides. For each $i = 1,...,n$, assume the interior angle at P_i has radian measure θ_i.
    a. If $n = 3$, then we know $\theta_1 + \theta_2 + \theta_3 = \pi$. For the general case $n \geq 3$, find a formula for $\sum_{i=1}^{n} \theta_i$. That is, give a formula for the sum of the measures of the interior angles of the simple polygon $P_1P_2 \cdots P_n$.
    b. Give a proof that your formula in part a is correct. You may use the formula for $n = 3$ in your proof.
4. Suppose, $n \geq 4$ is a natural number and $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, …, $P_n = (x_n, y_n)$ are points in the plane so that polygon $P_1P_2 \cdots P_n$ is a (counter-clockwise description of a) simple equilateral polygon with n sides having common side length L. We will refer to the polygon $P_1P_2 \cdots P_n$ as our *track*. For each $i = 1,...,n$, assume the interior angle at P_i has radian measure θ_i. Let $3 \leq m < n$ and suppose W is a regular polygon with *m* sides and common side length L. We will refer to the polygon W as our *wheel*.
    a. Give conditions guaranteeing that if W is placed in the interior of the track with one of its sides coincident with one side of the track, then W can roll around the inside of the track by continually rotating clockwise (in the manner described in problem 1), staying completely in the interior of the track, and touching the vertices of the track in their natural counter-clockwise order.
    b. Do the analog of part a, with the wheel on the exterior of the track.
    c. Give the total measure of the angle of clockwise rotation that the interior wheel makes as it rolls one time around the interior of the track.
    d. Give the total measure of the angle of counter-clockwise rotation that the exterior wheel makes as it rolls one time around the exterior of the track.
    e. Give the difference of your answer in part d and your answer is part c.
    f. Determining the total distance traveled by each of the vertices of the wheel as it rolls around the track.

5. (bring out your creative side…) Create an equilateral polygonal track with common side length L. Create different regular polygonal wheels, each having side length L that can roll around either the inside or outside of the track. Create an interesting picture created by the track, different positions of the wheels, and/or the curves of motion of selected points on the wheels, as the wheels roll around the track. Make sure you give a complete description of how your picture was created.
6. Consider the curved track shown below in red along with the circles shown in black (of equal radius). Assume the track has circumference C, and each wheel has radius R. Imagine the wheel on the inside of the track rotating clockwise so that it can roll around the inside of the track. Similarly, imagine the wheel on the outside of the track rotating counter-clockwise so that it can roll around the outside of the track. Each wheel goes around the track exactly one time.



   a. Explain how this problem can be approximated by the setting in problem 4.
   b. Give the total measure of the angle of clockwise rotation that the interior wheel makes as it rolls one time around the interior of the track, in terms of C and R. Justify your answer.
   c. Give the total measure of the angle of counter-clockwise rotation that the exterior wheel makes as it rolls one time around the exterior of the track, in terms of C and R. What is the justification for your answer?
   d. Give the total distance travelled by the centers of the wheels as they roll once around the track.