Visualizing Objects in Multidimensional Space

Project Problem

2014 Math Contest

University of Houston

Directions

Your work should be bound in a 3 tab folder or 3 ring binder. The first page should be a title page containing the school name and the names of the project team members. The second page (and third or more if needed) should contain the table of contents for the project write-up. The subsequent pages should contain well written solutions to exercises 1-9 (stated on the last few pages of this document). The final portion of your write-up should contain a description and URL for the YouTube video requested in exercise 10. Exercise 10 is worth 25% of the total grade on the project. Exercises 1-9 are equally weighted.

Note: Some of the problems will require the use of a computer, and possibly some programming.

All work will be evaluated based upon quality, clarity, precision and presentation.

Project solutions can be submitted in either of the following ways:

1. On the day of the contest between 8 and 9 am.
2. Via email to jmorgan@math.uh.edu by 8am on the day of the contest. In this case, the document should be attached as a pdf.

Email questions and comments, and check the contest homepage for updates.

How Do We Plot Objects in 3 Dimensional Space?

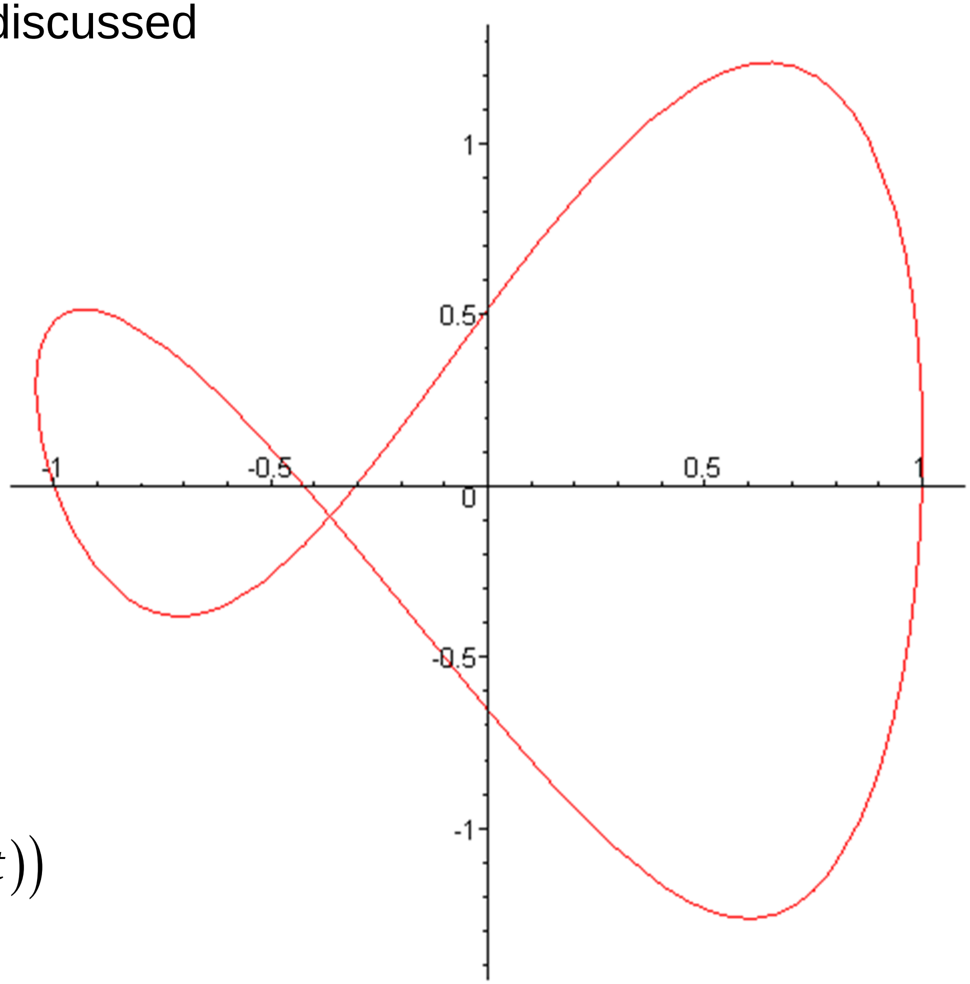
One approach is to use Orthogonal Projection onto a 2D view plane. We can illustrate visualization before we show how it is done by creating a rotating view of the curve given parametrically by

$$(\cos(t), \sin(t), \cos(2t))$$

See the next slide…

In this case, the View-Plane is Rotating

The axes show are axes for the view plane (to be discussed shortly).



Rotating the view of

$$\left(\cos(t), \sin(t), \cos(2t)\right)$$

You will have to play the slide show in Power Point to see the rotation.

Orthogonal Projection - Concepts

- Direction to the “eye” (only valid in 3 space)
- Orthogonal coordinate system on a view plane.
- Orthogonal projection onto the view plane.

Fundamental Concepts I

If n is a natural number, then R^n is the set of vectors v of the form

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

where each v_i is a real number. The vector v shown above can be visualized as a directed line segment from any given point $P = (p_1, p_2, \ldots, p_n)$ in n – space to the point $(p_1 + v_1, p_2 + v_2, \ldots, p_n + v_n)$ in n – space. Vectors are added and subtracted element by element, and vectors can be multiplied by real numbers (scalars) by multiplying each entry in the vector by the real number.

For example: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2.5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4.5 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 2.5 \end{pmatrix} = \begin{pmatrix} 4 \\ -0.5 \end{pmatrix}$, $2\begin{pmatrix} -3 \\ 2.5 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$

Note that R^n is closed under addition, subtraction and scalar multiplication.

Fundamental Concepts II

A vector in R^n is the zero vector if and only if all of its entries are 0, and it is a nonzero vector if and only if at least one of its entries is nonzero.

If n is a natural number, and u and v are vectors in R^n with

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ and } v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

then the dot product of u and v, denoted $u \cdot v$, is the real number given by

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

The Euclidean norm of v, denoted $|v|$, is the nonnegative real number given by

$$|v| = \sqrt{v \cdot v}$$

For example: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2.5 \end{pmatrix} = -3 + 5 = 2$ and $\left| \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right| = \sqrt{4 + 9} = \sqrt{13}$

Fundamental Concepts III

A vector u in R^n is said to be a unit vector if and only if $|u| = 1$. We can normalize a nonzero vector u in R^n by creating the unit vector $\frac{1}{|u|}u$.

You can verify that $|u \cdot v| \leq |u||v|$ for all vectors u and v in R^n, and this is the basis for defining the angle θ between two nonzero vectors u and v in R^n via the equation

$$\cos(\theta) = \frac{u \cdot v}{|u||v|}$$

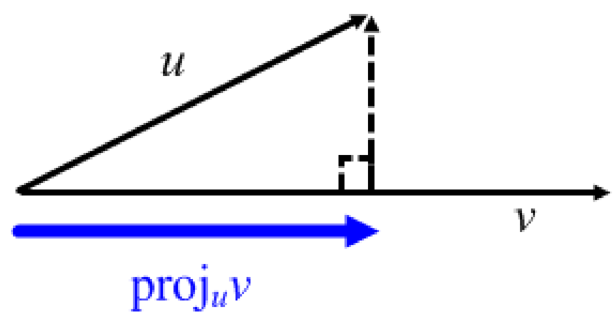
This motivates the definition that two vectors u and v in R^n are orthogonal (perpendicular) if and only if $u \cdot v = 0$. We say that u is orthogonal to v if and only if u and v are orthogonal.

Furthermore, u and v are orthonormal if and only if u and v are orthogonal unit vectors.

Fundamental Concepts IV

If u and v are vectors in R^n and v is not the zero vector, then we define the projection of u onto v, denoted $proj_v u$, by the vector

$$proj_v u = \frac{u \cdot v}{|v|^2} v$$



Fundamental Concepts V

If u is a vector in R^n with

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

then we write

$$[u_1 \quad u_2 \quad \cdots \quad u_n]^T = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

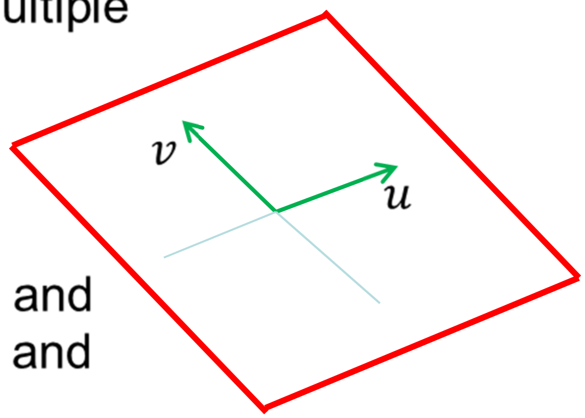
This notation is convenient for writing vectors "in line." For example, we can write $[2 \quad -1 \quad 0]^T$ instead of $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

Note: Points are visualized differently than vectors, but all of the operations associated with vectors can be performed on points.

View Plane With Orthogonal Coordinate System

A *view plane* in R^n is a 2 dimensional plane through the origin. That is, a view plane in R^n is a set S in R^n such that there are two nonzero vectors u and v in S such that neither is a scalar multiple of the other, and S is the set of all vectors of the form

$$\alpha u + \beta v$$



where α and β are real numbers. We say that the vectors u and v give an orthogonal coordinate system on S if and only if u and v are orthonormal.

A view plane in R^3 is just a plane through the origin. You should be able to show that if S is in R^3 and $[a \quad b \quad c]^T$ is a nonzero vector that is orthogonal to each of u and v, then the view plane can be visualized as the set of all points (x, y, z) where $ax + by + cz = 0$.

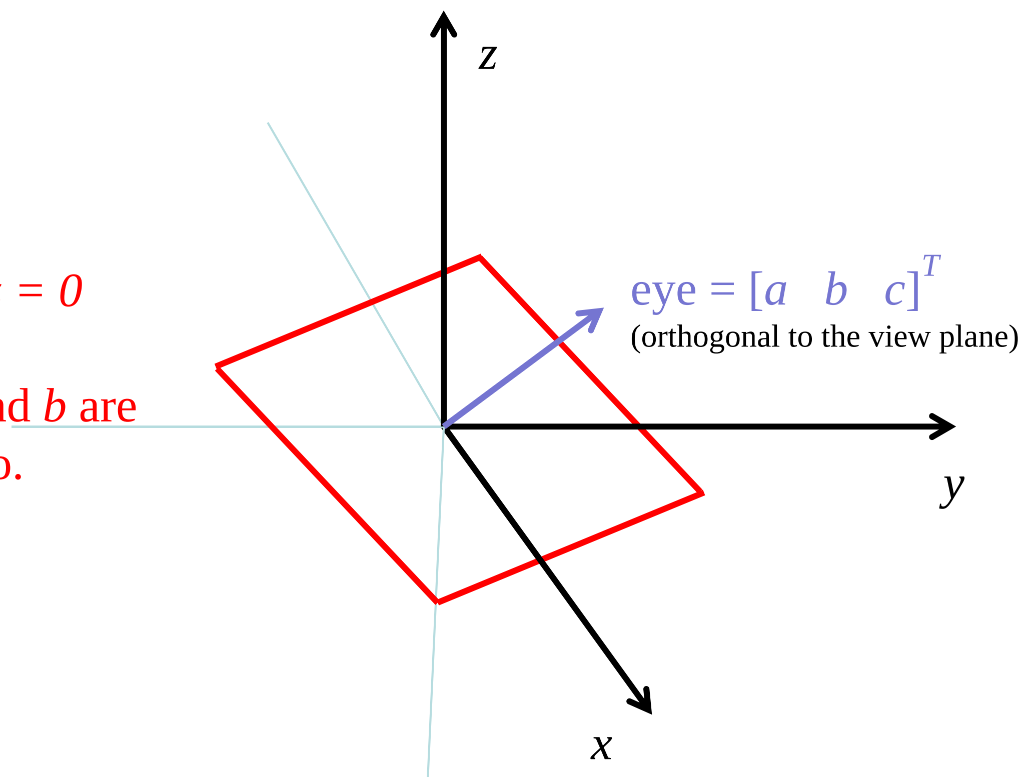
Orthogonal Projection

(3 Space Version)

View Plane

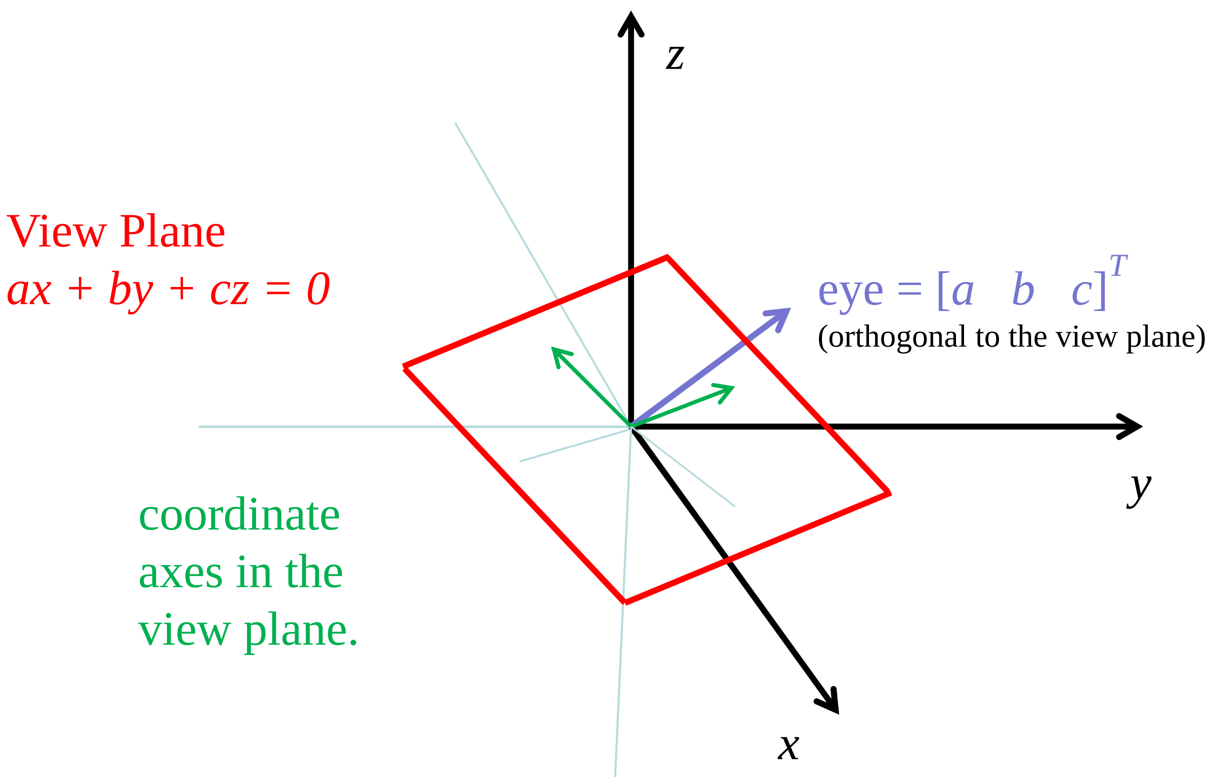
$ax + by + cz = 0$

Assume a and b are not both zero.



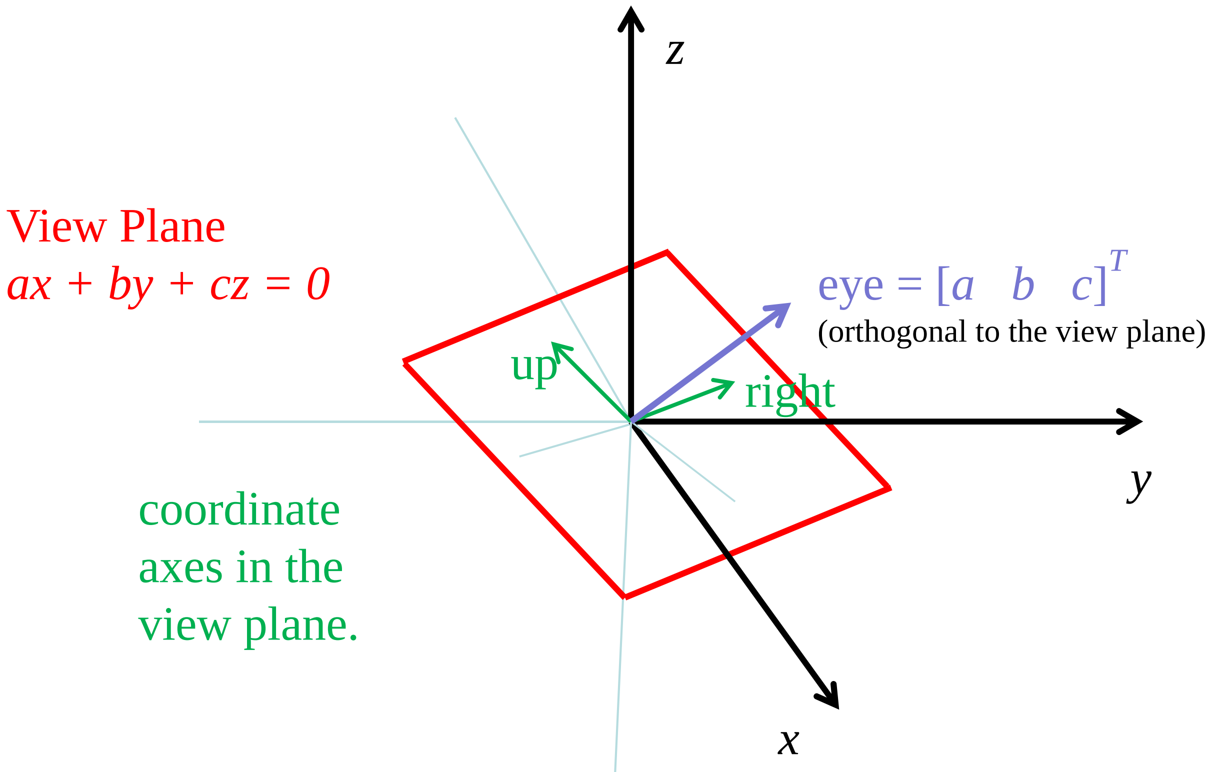
Orthogonal Projection

(3 Space Version)



Orthogonal Projection

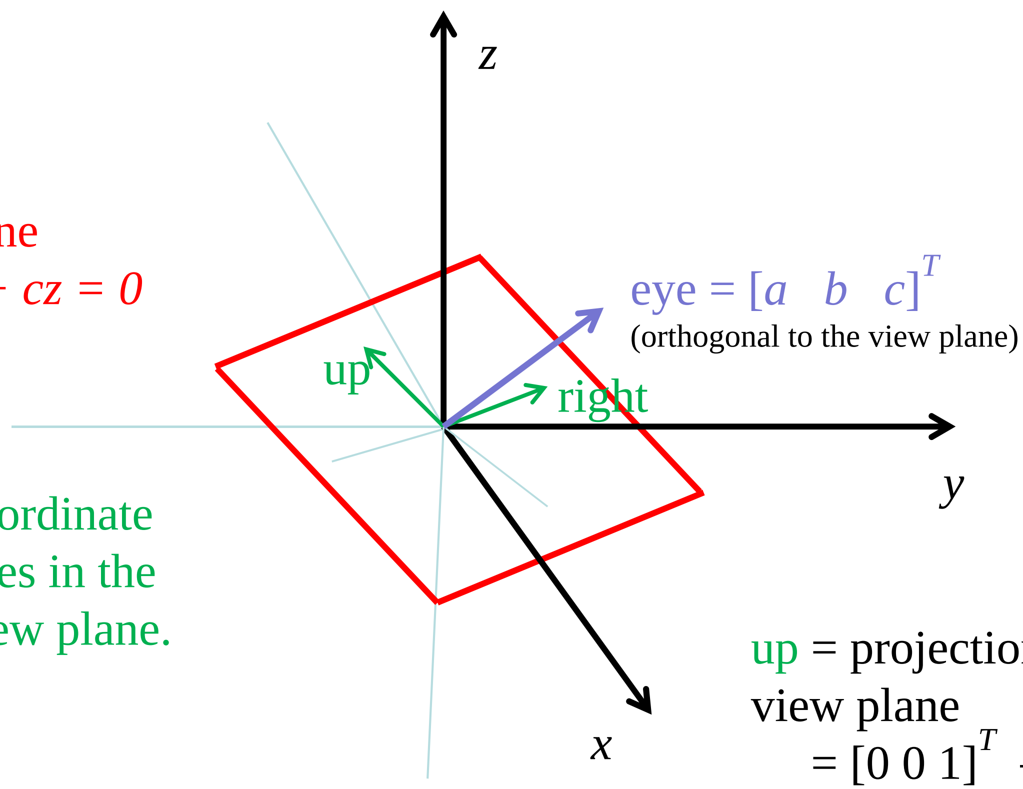
(3 Space Version)



Orthogonal Projection

(3 Space Version)

View Plane
$ax + by + cz = 0$



coordinate axes in the view plane.

up = projection of $[0\ 0\ 1]^T$ onto the view plane

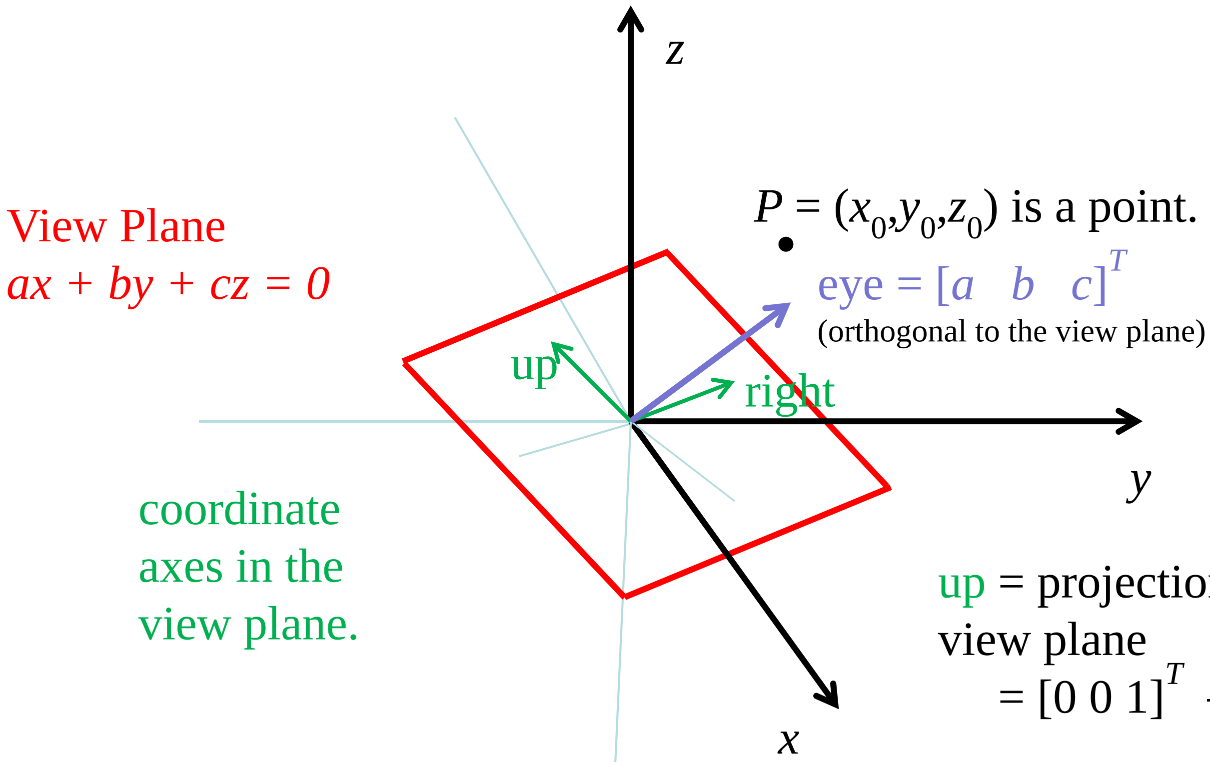
$$= [0\ 0\ 1]^T - \text{proj}_{\text{eye}}\, [0\ 0\ 1]^T$$

right = up x eye

Note: If u and v are in R^3, then $u \times v$ is the cross product of u and v, and $u \times v$ is orthogonal to each of u and v. Google “cross product.”

Orthogonal Projection

(3 Space Version)



up = projection of $[0\ 0\ 1]^T$ onto the view plane

$$= [0\ 0\ 1]^T - \text{proj}_{\text{eye}}\ [0\ 0\ 1]^T$$

right = up x eye

Then normalize these two vectors.

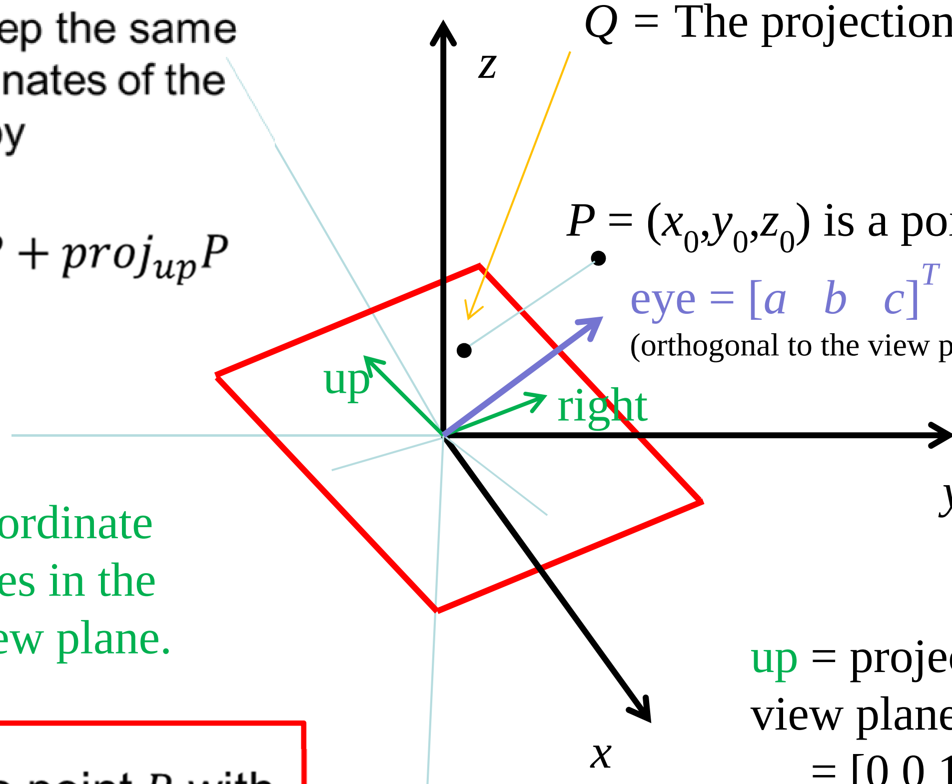
Note: If u and v are in R^3, then $u \times v$ is the cross product of u and v, and $u \times v$ is orthogonal to each of u and v. Google "cross product."

Orthogonal Projection

(3 Space Version)

Assume the normalized versions of up and $right$ keep the same names. The coordinates of the point Q are given by

$$Q = proj_{right}P + proj_{up}P$$



We visualize the point P with respect to the view plane by plotting the point

$$(P \cdot right, P \cdot up)$$

up = projection of $[0\ 0\ 1]^T$ onto the view plane

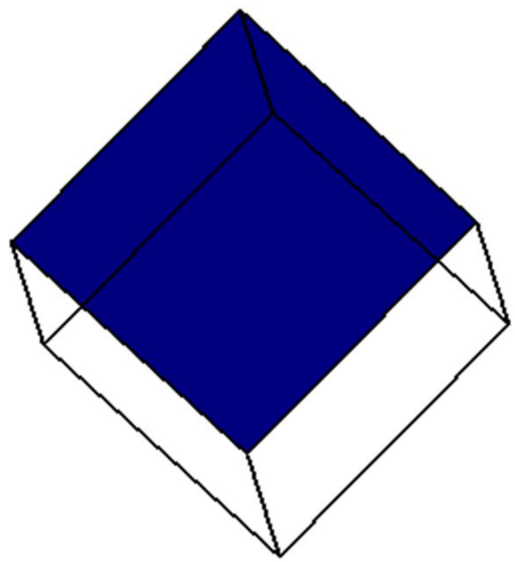
$= [0\ 0\ 1]^T - \text{proj}_{\text{eye}}\ [0\ 0\ 1]^T$

right = up x eye

Then normalize these to unit vectors.

Warning: Orthogonal projection can create illusions because there is no perspective.

Can you see the illusion in this rotating view?



You will have to play the slide show in Power Point
to see the rotation.

Matlab Code

(.m file to create a rotated view w/o face shading)

```
hold off

x=[0 0 0 0 0 NaN 1 0 NaN 1 0 NaN 1 0 NaN 1 0 NaN 1 1 1 1 1];
y=[0 0 1 1 0 NaN 0 0 NaN 1 1 NaN 1 1 NaN 0 0 NaN 0 0 1 1 0];
z=[0 1 1 0 0 NaN 0 0 NaN 0 0 NaN 1 1 NaN 1 1 NaN 0 1 1 0 0];

P=[x;y;z];
P=P-1/2;

plot([NaN],[NaN]);
axis([-2 2 -2 2]);
axis manual
axis off
ginput(1);
hold on
```

```
for i=0:2000
   theta=pi/4+i*pi/200;
   eye=[cos(theta) sin(theta) 1];
   up=[0 0 1];
   pup=up-dot(up,eye)/dot(eye,eye)*eye;
   right=cross(pup,eye);
   u=right/norm(right);
   v=pup/norm(pup);
   xp=u*P;
   yp=v*P;
   plot(xp,yp,'b');
   axis([-2 2 -2 2]);
   axis off
   drawnow
   hold off
end
```

Note: If you do not have Matlab, then you can use this as pseudo code to see how the rotation can be created. You do not need Matlab to complete this project.

Visualizing a Point in R^n *wrt* a View Plane

This idea can be extended to visualize points in higher dimensional spaces. The only changes that occur deal with the lack of cross product. In this setting, the orthogonal vectors that define the axes of the view plane are usually given from the beginning, without any care for a notion of an *up vector*.

In this setting we first obtain two orthonormal vectors u and v in R^n, and we use these vectors as an orthogonal coordinate system for the view plane generated by u and v. We visualize a point P in R^n with respect to this view plane by plotting the point $(P \cdot u, P \cdot v)$. Furthermore, if we want to visualize a line segment connecting points P and Q in R^n, with respect to this view plane, then we plot the line segment connecting the points $(P \cdot u, P \cdot v)$ and $(Q \cdot u, Q \cdot v)$.

The Standard
n Dimensional Unit Cube

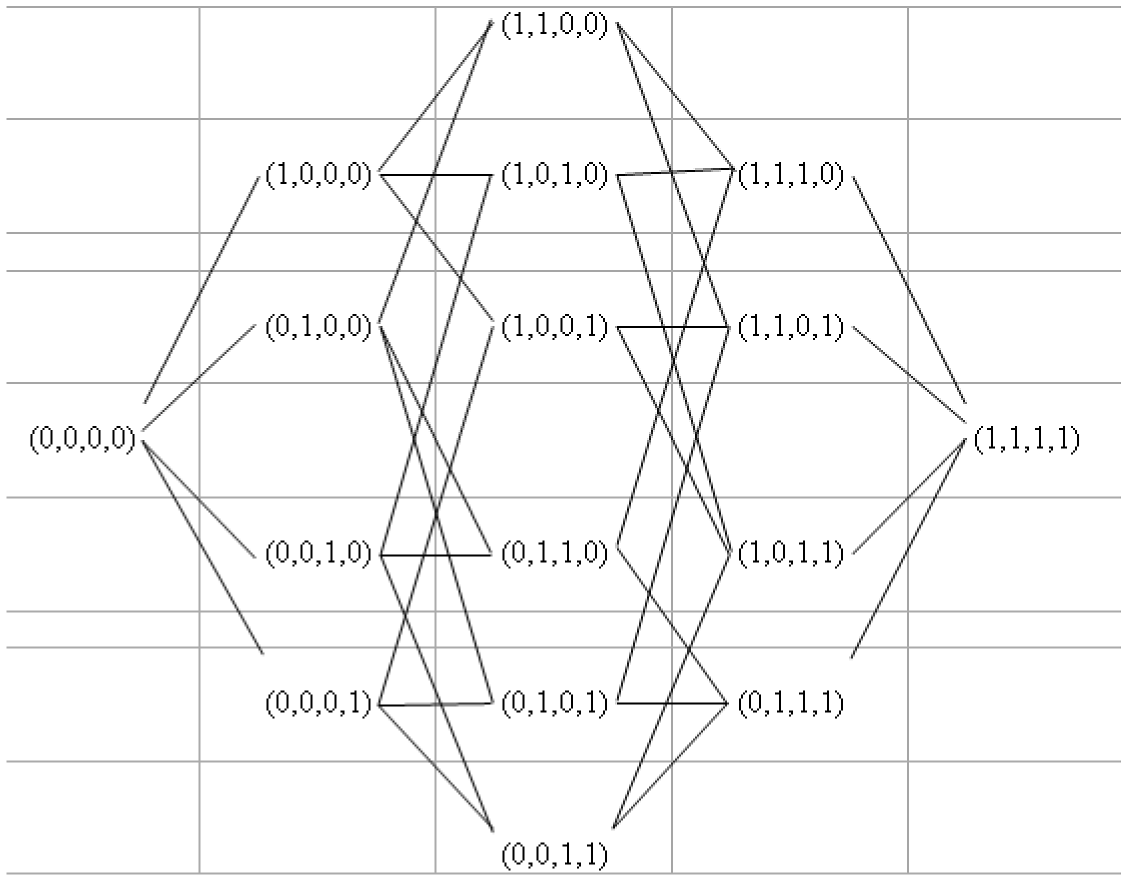
The standard n dimensional unit cube is the convex shape determined by the vertices $(e_1, \ldots, e_n)$, where each e_i is either 0 or 1. That is, if N is the total number of vertices, and the vertices are denoted by $P_1, \ldots, P_N$, then the standard n dimensional unit cube is the collection of all points given by

$$\alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_N P_N$$

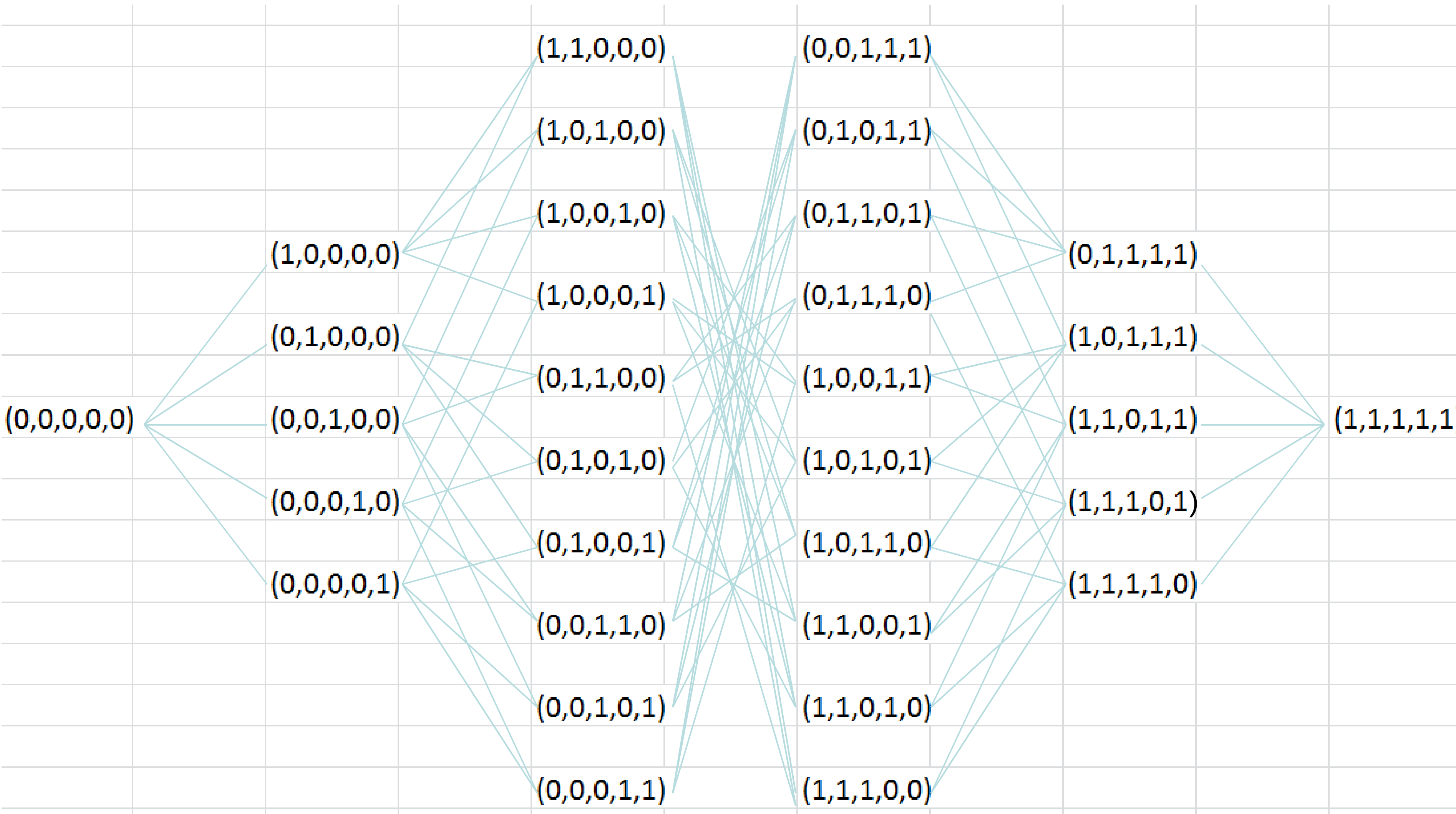
where each α_i is a nonnegative real number and $\alpha_1 + \cdots + \alpha_N \leq 1$.

The edges of the standard unit cube are the line segments connecting the vertices that differ from of each other in exactly one entry.

Standard 4 Dimensional Unit Cube Vertices and Edges



Standard 5 Dimensional Unit Cube Vertices and Edges



A Convenient View Plane

A convenient view plane in R^n is one that is generated by the vectors

$$u = \begin{pmatrix} cos(\pi/n) \\ cos(2\,\pi/n) \\ \vdots \\ cos(n\pi/n) \end{pmatrix} \text{ and } v = \begin{pmatrix} sin(\pi/n) \\ sin(2\,\pi/n) \\ \vdots \\ sin(n\pi/n) \end{pmatrix}$$

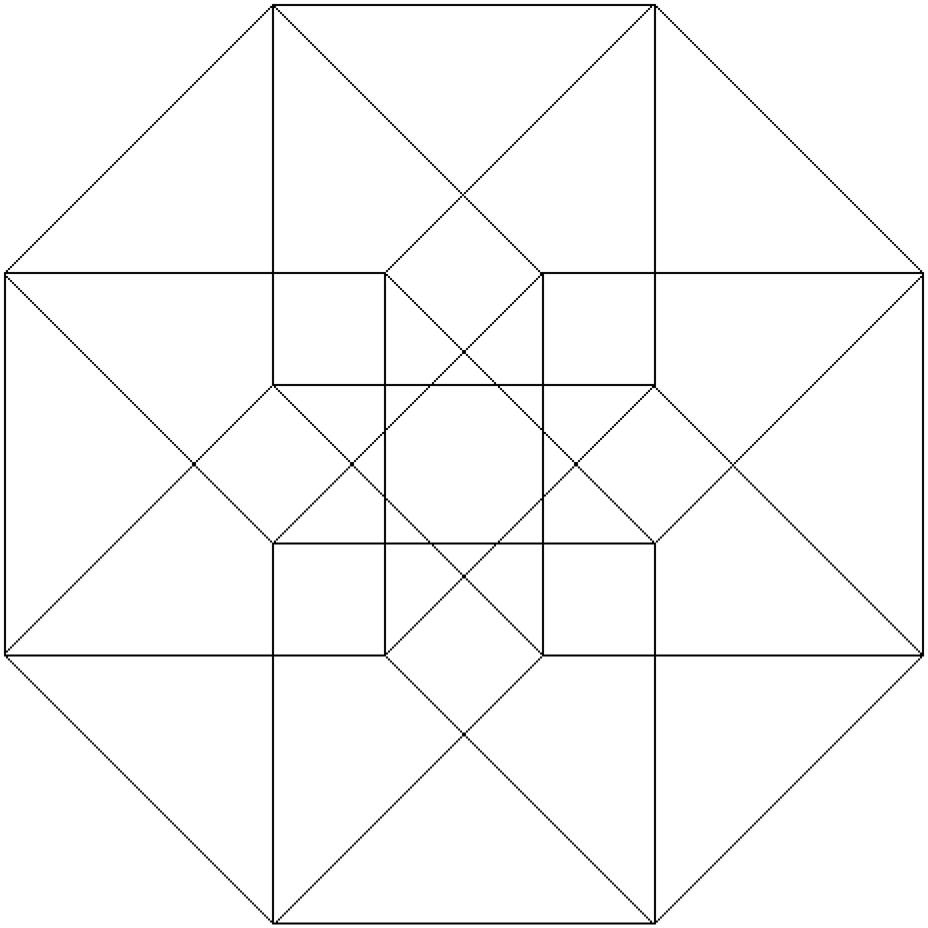
It is not difficult to show that u and v are orthogonal, and you will be asked to do this in one of the exercises.

Strategy for Visualizing the Standard *n* Dimensional Unit Cube

- Create orthonormal vectors that generate the view plane.
- Create the vertices for the cube.
- Create the coordinates for the projected vertices (in the view plane).
- Determine the connections to create edges.
- Sketch the edges by connecting the appropriate projected vertices in the view plane.

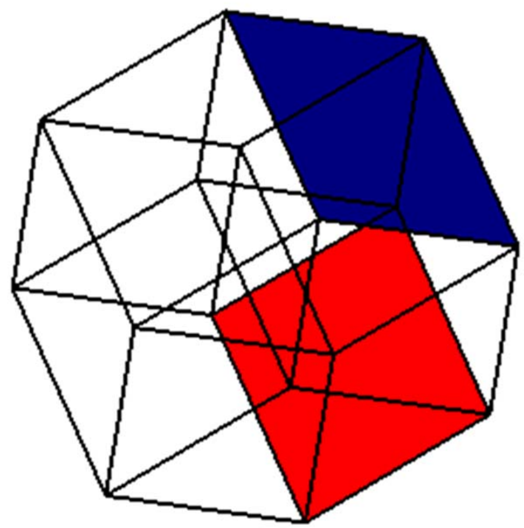
Note: The sketch created through this process only represents the edges of the cube, but it is enough to visualize the cube.

4 Dimensional Unit Cube *wrt* the Convenient View Plane



4 Dimensional Unit Cube with Rotation

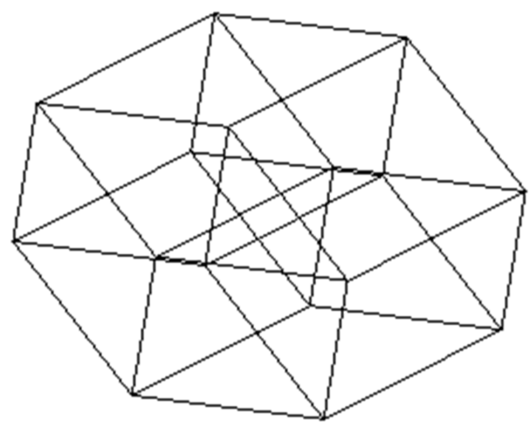
(actually, the view plane is rotating to create the effect)



I have colored 2 of the 2 dimensional faces in the cube so that you can see how they move with the rotation.

You will have to play the slide show in Power Point to see the rotation.

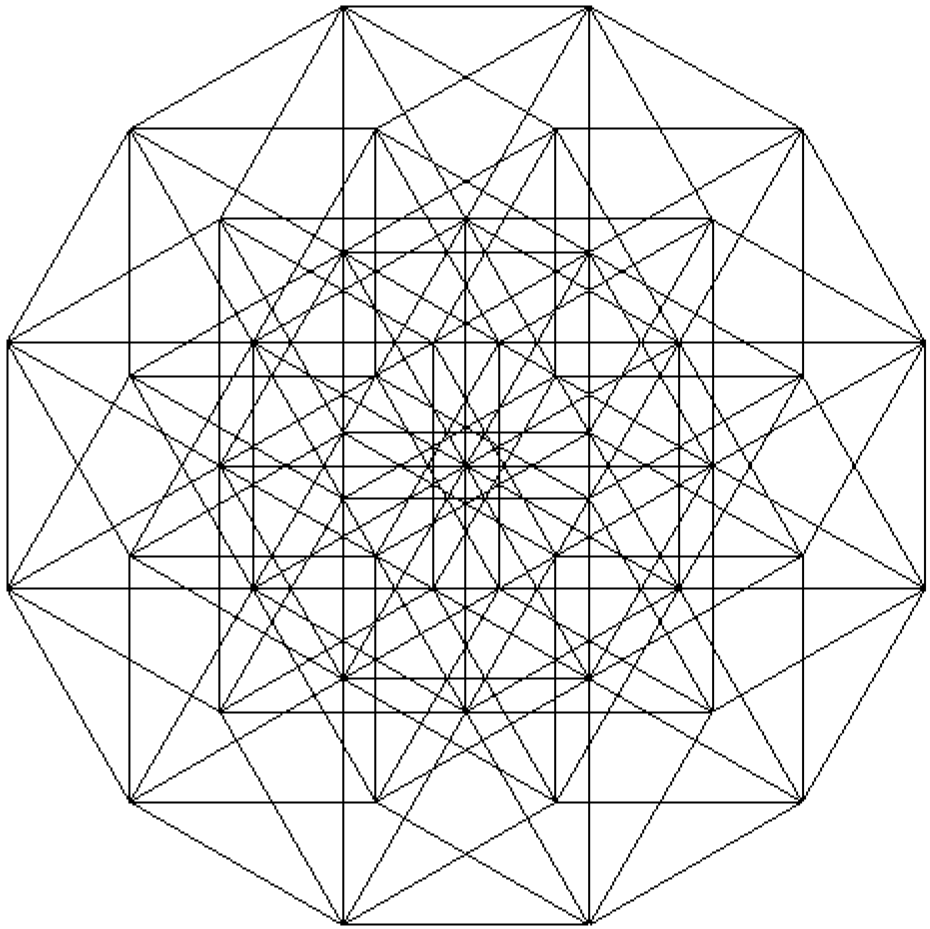
Another View of the Rotating 4 Dimensional Unit Cube



This one better illustrates changes in plots depending upon different view planes.

You will have to play the slide show in Power Point to see the rotation.

6 Dimensional Unit Cube *wrt* the Convenient View Plane



Exercises

1. Show that the vectors that generate the convenient view plane are orthogonal. Find $|u|$ and $|v|$. Are they orthonormal?
2. Use the convenient view plane in R^3 to visualize the curve given parametrically by $(\cos(t), \sin(t), \cos(2t))$.
3. Set $eye = [0.5 \quad 0.5 \quad 1]^T$, and create the view plane in R^3 that is perpendicular to eye. Then create the visualization of the standard 3 dimensional unit cube with respect to this view plane.
4. For each $\theta = i\pi/10$, for $i = 0, \ldots, 19$, set $eye = [\cos(\theta) \quad \sin(\theta) \quad 1]^T$, and create the view plane in R^3 that is perpendicular to eye. Then create the visualization of the standard 3 dimensional unit cube with respect to this view plane. Your work should produce 20 plots.
5. Produce the visualization of the 5 dimensional unit cube with respect to the convenient view plane in R^5.

   Then produce the visualization of the 7 dimensional unit cube with respect to the convenient view plane in R^7.

If you write a computer program to create the visualizations, then include your code.

Exercises Continued

6. Let C_n denote the standard n dimensional unit cube. The 1 dimensional faces of C_n are the edges of C_n. Notice that the edges are line segments that connect 2 vertices that differ in only one entry. More generally, we can define the notion of a k dimensional face of C_n when k is an integer between 1 and $n-1$. First, we fix $\mathrm{n}-k$ positions and values of either 0 or 1 for each of these positions, and choose all of the vertices in C_n that have these precise values in these positions. Suppose these vertices are given by $Q_1, \ldots, Q_M$. Then the associated k dimensional face is given by all points of the form $\alpha_1 Q_1 + \cdots + \alpha_M Q_M$ where each α_i is a nonnegative real number and $\alpha_1 + \cdots + \alpha_M \leq 1$. In addition, we can define the 0 dimensional faces of C_n to be the vertices of C_n. Give the number of k dimensional faces of C_n for each integer k between 0 and $n-1$.
7. Identify all of the 2 dimensional faces of C_3, and also describe the 2 and 3 dimensional faces of C_4. In general, describe how k dimensional faces of C_n are related to C_k.

Note: In problem 6, “between” includes the possibility of the first value and the last value.

Exercises Continued

8. Note that some of the k dimensional faces of C_n contain the origin. For each k, give the minimum number j, so that there are j distinct faces of dimension k containing the origin that only have the origin as a common point of intersection.
9. Suppose k and n are natural numbers. A set of vectors $\{v_1, \dots, v_k\}$ contained in R^n is said to be linearly independent if and only if the only collection of real numbers $\alpha_1, \dots, \alpha_k$ for which $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$ is the trivial collection where $\alpha_i = 0$ for all i. A subset S of R^n is said to be a k dimensional subspace of R^n if and only if there exists a linearly independent set of vectors $\{v_1, \dots, v_k\}$ so that S is the collection of all vectors of the form $\alpha_1 v_1 + \cdots + \alpha_k v_k$ where each α_i is a real number. For each possible k, and for each natural number j, determine whether every collection of j distinct subspaces of dimension k in R^n have a point other than the origin in common. Justify your answer.
10. (very important) Create a YouTube video of no more than 5 minutes in length where you creatively discuss n dimensional cubes.