**Calculus Exam**
**2009 University of Houston Math Contest**

**Name:** _____

**School:** _____

Please read the questions carefully and give a clear indication of your answer on each question.

There is no penalty for guessing.

Judges will use written comments and/or calculations to settle ties.
Good luck.

**NAME:** _____

**SCHOOL:** _____

**University of Houston**
**High School Contest – Spring 2009 Calculus Test**

1. What is $\lim_{h\to 0} \dfrac{\sin\left(\dfrac{3\pi}{4}+h\right)-\sin\left(\dfrac{3\pi}{4}\right)}{h}$?

   (a) $\sqrt{3}/2$
   (b) $-\sqrt{2}/2$
   (c) $-1/2$
   (d) $\sqrt{2}/2$
   (e) The limit does not exist

2. Let $f$ be some function for which you know only that

$$\text{if} \qquad 0<|x-4|<1, \qquad \text{then} \qquad |f(x)-5|<0.1.$$

   Which of the following statements are necessarily true?

   I. If $|x-4|<0.1$, then $|f(x)-5|<0.01$.

   II. If $|x-3.6|<0.3$, then $|f(x)-5|<0.1$.

   III. If $0<|x-4|<0.5$, then $|f(x)-5|<0.1$.

   IV. $\lim_{x\to 4} f(x)=5$.

   V. If $|x-4.6|<0.5$, then $|f(x)-5|<0.1$.

   (a) II and V
   (b) I and IV
   (c) I, III and V
   (d) II and III
   (e) II, III and IV

3. $\lim_{x \to 0^+} \tan\left(\dfrac{\sin 2\pi x}{6x}\right) =$

(a) $1/\sqrt{3}$

(b) $1$

(c) $\sqrt{3}$

(d) $\sqrt{2}/2$

(e) The limit does not exist

4. If $F'$ is a continuous function for all real $x$, then

$$\lim_{h \to 0} \frac{1}{h} \int_a^{a+h} F'(x)\, dx$$

is

(a) $0$

(b) $F(0)$

(c) $F(a)$

(d) $F'(a)$

(e) $F'(0)$

5. Set

$$g(x) = \begin{cases} a\sqrt{x+2} & 0 < x < 2, \\ bx + 2 & 2 \leq x < 5. \end{cases}$$

The values of $a$ and $b$ such that $g$ is differentiable on $(0, 5)$ are:

(a) $a = 4/3,\ b = 1/3$

(b) $a = 3,\ b = 2$

(c) $a = 7/5,\ b = 2/5$

(d) $a = 2,\ b = 1$

(e) $a = 8/5,\ b = 3/5$

6. If $\frac{d}{dx}f(x) = g(x)$ and if $h(x) = x^3$, then $\frac{d}{dx}f(h(x)) =$

(a) $g(x^3)$
(b) $3x^2\, g(x)$
(c) $x^3 g'(x)$
(d) $3x^2\, g(x^3)$
(e) $x^3 g(x^3)$

7. An equation for the normal line to the curve $y^3x + y^2x^2 = 6$ at the point $(2, 1)$ is

(a) $5x + 14y = 24$
(b) $14x - 5y = 33$
(c) $14x + 5y = -23$
(d) $5x - 14y = 4$
(e) $14x - 5y = 23$

8. Which of the following statements about the function $f(x) = x^4 - 3x^3$ is true?

(a) The graph of $f$ has no points of inflection and the function has one relative extremum.
(b) The graph of $f$ has one point of inflection and the function has two relative extrema.
(c) The graph of $f$ has two points of inflection and the function has one relative extremum.
(d) The graph of $f$ has two points of inflection and the function has two relative extrema.
(e) The graph of $f$ has one point of inflection and the function has no relative extrema.

9. A particle is moving along the $x$-axis so that its velocity at time $t$, $0 \le t \le 10$ is

$$v(t) = \ln\left(t^2 - 4t + 4\right).$$

During which time intervals is the particle moving to the left?

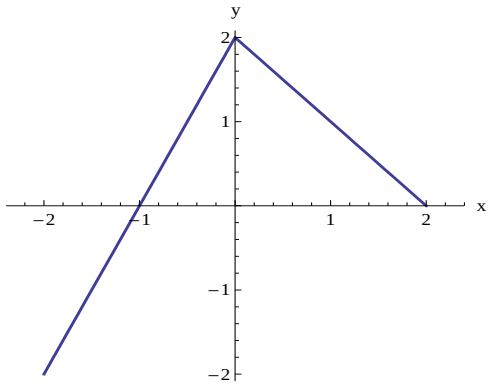
(a) $5 < t < 10$
(b) The particle never moves left.
(c) $1 < t < 3$
(d) $0 < t < 1$
(e) $1 < t < 5$

10. If $\int_0^4 f(x)\,dx = 5$, $\int_2^4 f(x)\,dx = 7$, and $\int_0^7 f(x)\,dx = 10$, then $\int_7^2 f(x)\,dx =$

(a) 8
(b) $-12$
(c) $-2$
(d) 12
(e) $-8$

11. The graph of the function $f$ shown below consists of two line segments. If $g(x) = \int_1^x f(t)\,dt$, then $g(-1) =$

(a) $-5/2$
(b) 3
(c) $5/2$
(d) $-2$
(e) $-3/2$



12. If $f$ is a continuous function and $F(x) = \int_0^x \left[ (2t+3) \int_t^2 f(u)\, du \right] dt$, then $F''(2) =$

(a) $-2f(2)$

(b) $-7f(2)$

(c) $7f'(2)$

(d) $3f'(2)$

(e) $7f(2)$

13. A curve in the plane is defined by the parametric equations: $x = e^{2t} + 2e^{-t}$, $y = e^{2t} + e^{t}$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $7x + 10y = -8$

(b) $5x - 6y = -11$

(c) $5x - 3y = 7$

(d) $10x - 7y = 8$

(e) $3x - 2y = 3$

14. Which of the following integrals represents the area enclosed by the inner loop of the graph of $r = 1 + 2\sin\theta$? Select all correct answers.

(a) $\frac{1}{2} \int_{3\pi/2}^{11\pi/6} (1 + 2\sin\theta)^2\, d\theta$

(b) $\frac{1}{2} \int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\, d\theta$

(c) $\frac{1}{2} \int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2\, d\theta$

(d) $\int_{7\pi/6}^{3\pi/2} (1 + 2\sin\theta)^2\, d\theta$

(e) $\frac{1}{2} \int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2\, d\theta$

15. The function $f(x) = 3 + \int_2^x \sqrt{5+2t}\,dt$ has an inverse. $(f^{-1})'(3) =$.

(a) 3
(b) $\sqrt{5}$
(c) 5
(d) $\sqrt{11}$
(e) 1/3

16. The values of $x$ for which the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2\, x^n}$ converges are

(a) All $x$ except $x = 0$
(b) $-1/2 \le x \le 1/2$
(c) $-2 < x < 2$
(d) $|x| \ge 2$
(e) $|x| > 1/2$

17. Suppose that the power series $\sum_{k=0}^{\infty} a_k(x-2)^k$ converges at $x = 5$. Which of the following series must be convergent?

I. $\sum_{k=0}^{\infty} a_k 4^k$
II. $\sum_{k=0}^{\infty} (-1)^k a_k 3^k$
III. $\sum_{k=0}^{\infty} (-1)^k a_k$
IV. $\sum_{k=0}^{\infty} a_k 2^k$

(a) I only
(b) II and IV
(c) III and IV
(d) I and III
(e) I, II and III

18. The region in the first quadrant bounded by the graph of $y = \cos(x^2)$, the vertical line $x = \sqrt{\pi}/2$, and the $x$-axis is revolved about the $y$-axis. The volume of the generated solid is:

(a) $2\pi$

(b) $\dfrac{\pi\sqrt{2}}{2}$

(c) $\pi$

(d) $\pi\sqrt{2}$

(e) $\pi/2$

19. The length of the graph of $f(x) = \ln \sec x,\ 0 \le x \le \pi/3$ is:

(a) $3 + \sqrt{2}$

(b) $\ln\left(2 + \sqrt{3}\right)$

(c) $\ln\left(\sqrt{3}\right)$

(d) $2 + \sqrt{3}$

(e) $\ln\left(\dfrac{1 + \sqrt{3}}{2}\right)$

20. If the function $f(x) = x^3 + 1$ has an average value of $9$ on the interval $[0, k]$, then $k =$

(a) $9/4$

(b) $18^{1/3}$

(c) $32^{1/4}$

(d) $36^{1/3}$

(e) $32^{1/3}$

21. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are necessarily true?

I. If $a_n > 0$ for all $n$ and $a_n \to L$, then $L > 0$.

II. If $\{a_n\}$ is not bounded below, then it diverges.

III. If $a_n \geq 0$ for all $n$ and $a_n \to L$, then $L \geq 0$.

IV. If $\{a_n\}$ is nondecreasing and unbounded above, then it converges.

(a) II and III
(b) III only
(c) II and IV
(d) II, III and IV
(e) I and II

22. $\lim_{x\to\infty} e^{-x^2} \int_0^x 2te^{t^2}\, dt =$

(a) 0
(b) 1
(c) 2
(d) $e$
(e) $e^2$.

23. $\int_0^3 \frac{1}{(1-x)^2}\, dx =$

(a) $-3/2$
(b) $-1/2$
(c) $1/2$
(d) $3/2$
(e) divergent

24. If $\dfrac{dy}{dx} = y \sec^2 x$ and $y(0) = 5$, then $y =$

(a) $\tan x + 5$

(b) $e^{\tan x} + 4$

(c) $5e^{\tan x}$

(d) $e^{\tan x} + 5$

(e) $\tan x + 5e^x$

25. The function $f$ is infinitely differentiable, $f(0) = 4$, and

$$f^{(n)}(0) = \frac{(n-1)!}{3^n} \quad \text{for all } n \geq 1.$$

The interval of convergence of the Taylor series for $f$ in powers of $x$ is:

(a) $-3 < x < 3$

(b) $-1 \leq x < 1$

(c) $-1/3 \leq x \leq 1/3$

(d) $-3 \leq x < 3$

(e) $-3 < x \leq 3$

**Tie Breaker**

26. An athlete is running around a circular track of radius 100 meters at the rate of 5 meters/second. A spectator is 300 meters from the center of the track. How fast is the distance between the runner and the spectator changing when the runner is approaching the spectator and the distance between them is 250 meters? Show your work