Name: ______________________________

University of Houston
High School Math Contest – Spring 2011 Calculus Test

NAME: ______________________________

SCHOOL: ______________________________

1. $\lim_{x \to 0} \sec\left(\dfrac{\sin 2\pi x}{3x}\right) =$

   (a) 2
   (b) $-2/\sqrt{3}$
   (c) -2
   (d) $\sqrt{3}$
   (e) The limit does not exist

2. Find the numbers a such that $\lim_{x \to 0} \dfrac{e^{ax^2} - \cos 2x}{x^2} = 8.$

   (a) $a = 3$
   (b) $a = 6$
   (c) $a = 4$
   (d) $a = 2$
   (e) $a = 1$

3. A function f is defined on an interval $[a, b]$. Which of the following statements are always true?

   I. If f is differentiable on (a, b), and $f(a)$ and $f(b)$ have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   II. If f is continuous on $[a, b]$, and $f(a)$ and $f(b)$ have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   III. If f is continuous on $[a, b]$ and there is a point c in (a, b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

   IV. If f is differentiable on (a, b) and has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

   (a) I, III, IV
   (b) I, II
   (c) III only
   (d) II, IV
   (e) II only

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4. The values of A and B such that

$$f(x) = \begin{cases} Ax^3 + Bx + 2, & x \le 2 \\ Bx^2 - A, & x > 2 \end{cases}$$

is everywhere differentiable are:

(a) $A = -2,\ B = -8$
(b) $A = 2,\ B = 10$
(c) $A = 2,\ B = 8$
(d) $A = 1,\ B = 4$
(e) $A = -2,\ B = 10$

5. $\lim\limits_{h \to 0} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}+h} \dfrac{\sin x}{x}\, dx =$

(a) 0
(b) $\dfrac{\sqrt{2}}{2}$
(c) 1
(d) $\dfrac{2\sqrt{2}}{\pi}$
(e) $\dfrac{\sqrt{2}}{2\pi}$

6. Suppose that f is continuous on $[1, 5]$ and differentiable on $(1, 5)$. Suppose also that $f(1) = 3$ and $f(5) = -1$. Which of the following statements is not necessarily true?

(a) The Mean-Value Theorem applies to f.
(b) f is integrable on $[1, 5]$.
(c) There exists a number $c \in (1, 5)$ such that $f'(c) = 1$.
(d) If k is a number between -1 and 3, then there exists a number $c \in (1, 5)$ such that $f(c) = k$.
(e) If c is any number such that $1 < c < 5$, then $\lim\limits_{x \to c} f(x)$ exists.

7. If $f'(x) = h(x)$ and $g(x) = x^3 + 1$, then $\dfrac{d}{dx} f[g(x)] =$

(a) $h(x^3 + 1)$
(b) $3x^2 h(x^3 + 1)$
(c) $3x^2 h(x)$
(d) $3x^2 h'(x^3 + 1)$
(e) $(x^3 + 1)h(3x^2)$

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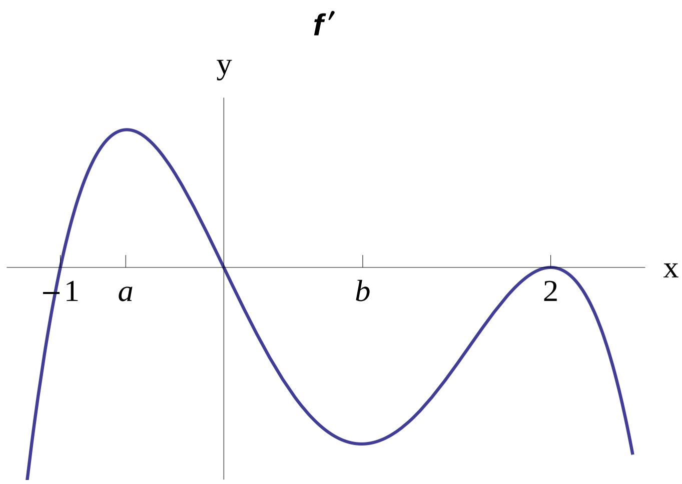
8. An equation for the normal line to the curve $2x^3 + 2y^2 = 5xy$ at the point $(1, 2)$ is

(a) $3x - 4y = 2$
(b) $3x + 4y = -5$
(c) $3x + 4y = 11$
(d) $4x - 3y = -2$
(e) $4x + 3y = 10$

9. The graph of the derivative of a function f is shown below.



Which of the following is (are) not true? Select all correct answers.

(a) f has a local maximum at $x = 0$.
(b) f is increasing on $[-1, 0]$.
(c) f has a point of inflection at $x = 2$.
(d) f is concave up on $[0, 2]$.
(e) f is concave down on (a, b).

10. A rectangle with one side on the x-axis is inscribed in the triangle formed by the lines $y = x$, $y = 0$, and $2x + y = 12$. The maximum area of such a rectangle is:

(a) 6
(b) 3
(c) 5/2
(d) 5
(e) 7

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11. Set $f(x) = \dfrac{4}{1+x^2}$, and let $H(x) = \int_0^x f(t)\,dt$. The local linearization of H at $x = 1$ is

(a) $y = 2x$
(b) $y = 2x + \pi - 2$
(c) $y = -2x - 4$
(d) $y = 2x + \pi$
(e) $-2x + 2\ln 2$

12. Given that $\int_0^2 f(x)\,dx = \dfrac{8}{3}$, $\int_1^2 f(x)\,dx = \dfrac{4}{3}$, and $\int_0^3 f(x)\,dx = \dfrac{11}{3}$, find $\int_3^1 f(x)\,dx$.

(a) $-5/3$
(b) 2
(c) $7/3$
(d) $5/3$
(e) $-7/3$

13. The region bounded by $y = e^x$, $y = 1$, and the line $x = 2$ is rotated about the x-axis. Which of the following integrals gives the volume of the solid which is generated:

$$(A)\ \pi\int_0^2 e^{2x}\,dx, \qquad (B)\ 2\pi\int_1^{e^2} y(2-\ln y)\,dy, \qquad (C)\ \pi\int_0^2 \left(e^{2x}-1\right)\,dx$$

$$(D)\ 2\pi\int_0^{e^2} y(2-\ln y)\,dy, \qquad (E)\ \pi\int_0^2 (e^x-1)^2\,dx$$

(a) (A) and (D)
(b) (B) and (C),
(c) (A), (D) and (E)
(d) (C) and (D)
(e) (B) only

14. If f is a continuous function and $F(x) = \int_0^x \left[(t^2+1)\int_2^t f(u)\,du\right] dt$, then $F''(2) =$

(a) $5f(2)$
(b) 5
(c) $4f(2)$
(d) $5f'(2)$
(e) $4f'(2)$

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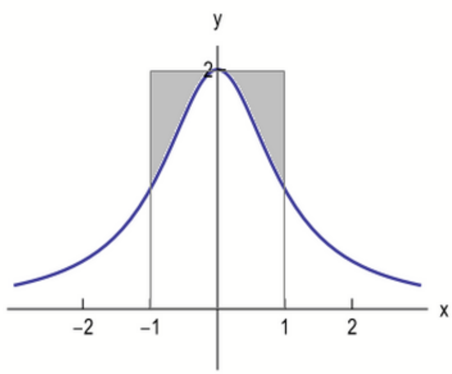
15. The work done in lifting an object is the product of the weight of the object and the distance it is moved. A cylindrical barrel 2 feet in diameter and 4 feet high is half full of oil weighing 50 pounds per cubic foot. The work done, in foot-pounds, in pumping the oil to the top of the container is:

(a) 100π
(b) 200π
(c) 300π
(d) 600π
(e) 1200π

16. The curve in the figure shown below is given by $y = \dfrac{2}{1+x^2}$. Find the area of the shaded region.

(a) $4 - \dfrac{\pi}{2}$
(b) $4 - \pi$
(c) $4 - 2\pi$
(d) $4 - \dfrac{\pi}{4}$
(e) $2\pi - 4$



17. The function $f(x) = x^3 + 2x - 9$ has an inverse. If the graph of f passes through the point $(2,3)$, then $(f^{-1})'(3) =$

(a) 14
(b) 1/29
(c) 1/12
(d) 1/14
(e) 29

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18. If $x = 2\sin\theta$, then $\displaystyle\int_1^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$ is equivalent to

(a) $4\displaystyle\int_1^2 \sin^2\theta\,d\theta$

(b) $2\displaystyle\int_0^{\pi/2} \sin\theta\,\tan\theta\,d\theta$

(c) $2\displaystyle\int_{\pi/6}^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

(d) $4\displaystyle\int_0^{\pi/2} \sin^2\theta\,d\theta$

(e) $4\displaystyle\int_{\pi/6}^{\pi/2} \sin^2\theta\,d\theta$

19. The general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{1-2x}{y}$ is a family of:

(a) straight lines

(b) circles

(c) ellipses

(d) parabolas

(e) hyperbolas

20. Which differential equation which has the slope field

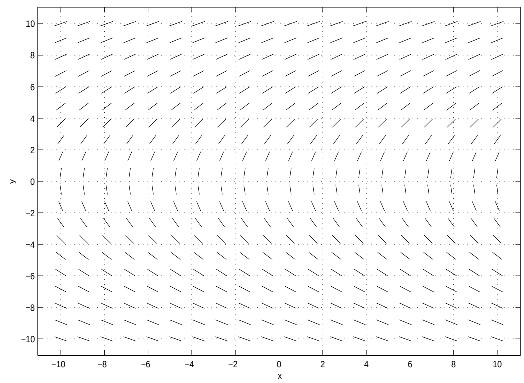
(a) $\dfrac{dy}{dx} = \dfrac{5}{y}$

(b) $\dfrac{dy}{dx} = \dfrac{5}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(d) $\dfrac{dy}{dx} = 5y$

(e) $\dfrac{dy}{dx} = x + y$



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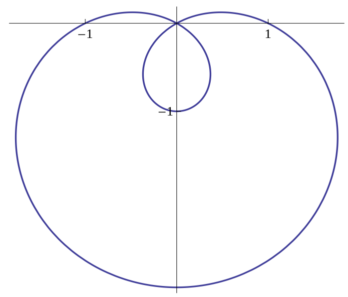
21. The curve $r = 1 - 2\sin\theta$ is shown in the figure. The area enclosed by the inner loop of the curve is:



(a) $\pi - \dfrac{3\sqrt{3}}{2}$

(b) $\pi - 3\sqrt{3}$

(c) $3\sqrt{3} - \pi$

(d) $\pi + \dfrac{3\sqrt{3}}{2}$

(e) $\dfrac{1}{2}\left(3\pi - 3\sqrt{3}\right)$

22. The length of the graph of $f(x) = \ln \sec x,\ 0 \leq x \leq \pi/3$ is:

(a) $3 + \sqrt{2}$

(b) $\ln\left(2 + \sqrt{3}\right)$

(c) $\ln\left(\sqrt{3}\right)$

(d) $2 + \sqrt{3}$

(e) $\ln\left(\dfrac{1 + \sqrt{3}}{2}\right)$

23. A particle moves along the parabola $x = 3y - y^2$ so that $dy/dt = 3$ at all times t. The speed of the particle when it is at the point $(2, 1)$ is:

(a) $2\sqrt{2}$

(b) 3

(c) $\sqrt{13}$

(d) $3\sqrt{2}$

(e) $2\sqrt{3}$

24. If 40 grams of a radioactive substance decays to 20 grams in two years, then, to the nearest gram, the amount left after 3 years is:

(a) 11

(b) 12

(c) 14

(d) 16

(e) 17

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25. A curve in the plane is defined by the parametric equations: $x = e^{2t} + 2e^{-t},\ y = e^{2t} + e^{t}$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $10x - 7y = 8$
(b) $5x - 6y = -11$
(c) $5x - 3y = 7$
(d) $7x + 10y = -8$
(e) $3x - 2y = 3$

26. If a block of ice melts at the rate of $\dfrac{72}{2t+3}$ cm^3/min, then the closest approximation to the amount of ice which melts during the first three minutes is:

(a) $16\,\text{cm}^3$
(b) $22\,\text{cm}^3$
(c) $31\,\text{cm}^3$
(d) $40\,\text{cm}^3$
(e) $68\,\text{cm}^3$

27. Which infinite series converge(s)?

(I) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$ (II) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$ (III) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3}{n^4+1}$

(a) I only
(b) I and II
(c) II only
(d) II and III
(e) I, II and III

28. The radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ is:

(a) 0
(b) $1/e$
(c) 1
(d) e
(e) ∞

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29. The function f is infinitely differentiable, $f(2) = 4$, and

$$f^{(n)}(2) = \frac{(n-1)!}{3^n} \quad \text{for all} \quad n \geq 1.$$

The interval of convergence of the Taylor series for f is:

(a) $-\infty < x < \infty$
(b) $-3 < x < 3$
(c) $-1 < x < 5$
(d) $0 \leq x \leq 4$
(e) $-1 \leq x < 5$

30. Using two terms of an appropriate Maclaurin series, estimate $\displaystyle\int_0^1 \frac{1-\cos x}{x}\,dx$.

(a) $\dfrac{19}{96}$
(b) $\dfrac{23}{96}$
(c) $\dfrac{1}{4}$
(d) $\dfrac{25}{96}$
(e) undefined; the integral is improper