University of Houston
High School Math Contest – Spring 2012 Calculus Test

NAME: ______________________________

SCHOOL: ______________________________

1. $\lim_{x\to 0} \tan\left(\dfrac{2\sin 3\pi x}{9x}\right) =$

   (a) $1/\sqrt{3}$
   (b) $-\sqrt{3}$
   (c) $-1/\sqrt{3}$
   (d) $\sqrt{3}$
   (e) The limit does not exist

2. Let f be some function for which you know only that

$$\text{if} \qquad 0 < |x-3| < 1, \qquad \text{then} \qquad |f(x) - 4| < 0.1.$$

   Which of the following statements are necessarily true?

   I. If $|x-3| < 0.1$, then $|f(x) - 4| < 0.01$.

   II. If $|x-2.6| < 0.3$, then $|f(x) - 4| < 0.1$.

   III. If $0 < |x-3| < 0.5$, then $|f(x) - 4| < 0.1$.

   IV. $\lim_{x\to 3} f(x) = 4$.

   (a) II and IV
   (b) IV only
   (c) I and II
   (d) II and III
   (e) II, III and IV

3. What is $\lim_{h\to 0} \dfrac{\cos\left(\dfrac{5\pi}{6} + h\right) - \cos\left(\dfrac{5\pi}{6}\right)}{h}$?

   (a) $1/2$
   (b) $\sqrt{3}/2$
   (c) $-1/2$
   (d) $-\sqrt{3}/2$
   (e) The limit does not exist

4. A function f is defined on an interval $[a, b]$. Which of the following statements could be false?

   I. If f is differentiable on (a, b), and if $f(a)$ and $f(b)$, have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   II. If f is continuous on $[a, b]$, and if $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   III. If f is continuous on $[a, b]$ and there is a point c in (a, b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

   IV. If f is differentiable on (a, b) and if f has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

   (a) II only
   (b) II, III
   (c) I, III, IV
   (d) II, IV
   (e) I, III

5. Set

$$g(x) = \begin{cases} a\sqrt{x+1} & 0 < x < 3, \\ bx + 2 & 3 \leq x < 5. \end{cases}$$

   The values of a and b such that g is differentiable on $(0, 5)$ are:

   (a) $a = 4,\ b = 2$
   (b) $a = -8/5,\ b = -2/5$
   (c) $a = 2,\ b = 3/2$
   (d) $a = -1,\ b = -3/4$
   (e) $a = 8/5,\ b = 2/5$

6. When the local linearization of $f(x) = \sqrt{4 + \ln(1 + x)}$ near $x = 0$ is used, an estimate of $f(0.08)$ is:

   (a) 1.98
   (b) 2.01
   (c) 2.02
   (d) 2.04
   (e) 2.06

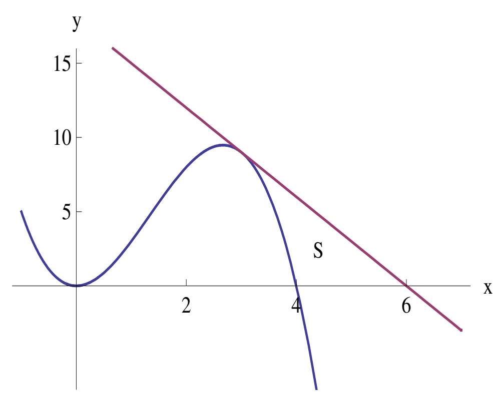
7. An equation for the normal line to the curve $2x^3 + 2y^2 = 5xy$ at the point $(1, 2)$ is

(a) $3x + 4y = 11$
(b) $3x + 4y = -5$
(c) $3x - 4y = 2$
(d) $4x - 3y = -2$
(e) $4x + 3y = 10$

8. Suppose that f is continuous on $[1, 5]$ and differentiable on $(1, 5)$. Suppose also that $f(1) = 3$ and $f(5) = -1$. Which of the following statements is not necessarily true?

(a) The Mean-Value Theorem applies to f.
(b) f is integrable on $[1, 5]$.
(c) There exists a number $c \in (1, 5)$ such that $f'(c) = 1$.
(d) If k is a number between -1 and 3, then there exists a number $c \in (1, 5)$ such that $f(c) = k$.
(e) If c is any number such that $1 < c < 5$, then $\lim_{x \to c} f(x)$ exists.

9. Set $f(x) = 4x^2 - x^3$, and let $\mathcal{L}$ be the line $y = 18 - 3x$, where $\mathcal{L}$ is tangent to the graph of f. Let S be the region bounded by the graph of f, the line $\mathcal{L}$ and the x-axis. The area of S is:



(a) $103/12$
(b) $43/6$
(c) $101/12$
(d) $95/12$
(e) $53/6$

10. A particle is moving along the x-axis so that its velocity at time t, $0 \le t \le 10$ is

$$v(t) = \ln\left(t^2 - 3t + 3\right).$$

During which time intervals is the particle moving to the left?

(a) $2 < t \le 10$
(b) $1 < t < 2$
(c) The particle never moves left.
(d) $0 \le t < 1$
(e) $0 < t < 5$

11. The maximum value of the function $f(x) = \dfrac{\ln x}{x}$ is

(a) e
(b) 1
(c) $1/e$
(d) $\sqrt{e}$
(e) None of these

12. Let R be the region bounded by the graph of $f(x) = \sqrt{x-1}$, the vertical line $x = 10$, and the x-axis. Find the volume of the solid generated when R is revolved about the line $y = 3$.

(a) $\dfrac{135\pi}{2}$
(b) $\dfrac{99\pi}{2}$
(c) $\dfrac{189\pi}{2}$
(d) $\dfrac{119\pi}{2}$
(e) $\dfrac{137\pi}{2}$

13. The region in the first quadrant bounded by the graph of $y = e^{2x}$, the vertical line $x = \ln 3$, and the x-axis is revolved about the y-axis. Find the volume of the solid that is generated.

(a) $2\pi\,[3\ln 3 - 2]$
(b) $\pi\,[9\ln 3 + 4]$
(c) $9\pi/2$
(d) $\pi\,[9\ln 3 - 4]$
(e) $\pi\left[\frac{5}{2}\ln 3 - \frac{1}{2}\right]$

14. If $\int_1^4 f(x)\,dx = 5$, $\int_3^4 f(x)\,dx = 7$, and $\int_1^8 f(x)\,dx = 11$, then $\int_8^3 f(x)\,dx =$.

(a) -9
(b) 13
(c) -1
(d) 9
(e) -13

15. If f is a continuous function and $F(x) = \int_0^x \left[(2t+3)\int_t^2 f(u)\,du\right] dt$, then $F''(2) =$

(a) $-2f(2)$
(b) $-7f(2)$
(c) $7f'(2)$
(d) $3f'(2)$
(e) $7f(2)$

16. A curve in the plane is defined by the parametric equations: $x = e^{2t} + 2e^{-t}$, $y = e^{2t} + e^t$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $7x + 10y = -8$
(b) $5x - 6y = -11$
(c) $5x - 3y = 7$
(d) $10x - 7y = 8$
(e) $3x - 2y = 3$

17. The function $f(x) = x^3 - \dfrac{2}{x} - 3$, $x \in (0, \infty)$, has an inverse. The graph of f passes through $(2, 4)$. $\left(f^{-1}\right)'(4) =$

(a) $25/2$
(b) $-2/27$
(c) $-4/25$
(d) $27/4$
(e) $2/25$

18. A point (x, y) is moving along a curve $y = f(x)$. At the instant when the slope of the curve is $-3/5$, the x-coordinate of the point is decreasing at the rate of 4 units per second. The rate of change, in units per second, of the y-coordinate is

(a) $12/5$
(b) $3/20$
(c) $-12/5$
(d) $5/12$
(e) $-20/3$

19. A curve in the plane is defined by the parametric equations: $x = 3t^2 + 2,\ y = \frac{2}{3}t^3 + 1,\ t \in [0, 4]$. Find the length of the curve.

(a) 244/3
(b) 98
(c) 196/3
(d) 250/3
(e) 196

20. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are necessarily true?

I. If $a_n > 0$ for all n and $a_n \to L$, then $L > 0$.

II. If $\{a_n\}$ is not bounded below, then it diverges.

III. If a_n converges, then it is bounded above.

IV. If $\{a_n\}$ is bounded, then it converges.

(a) III only
(b) I, III
(c) II, IV
(d) II, III, IV
(e) II, III

21. Evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$.

(a) $\pi/2$
(b) 2
(c) 1
(d) $\pi/4$
(e) divergent

22. The area of the region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$ is given by

(a) $\displaystyle\int_{\pi/6}^{\pi/2} (2\sin\theta - 1)^2\, d\theta$

(b) $\displaystyle\int_{\pi/6}^{\pi/2} \left[9\sin^2\theta - (1+\sin\theta)^2\right]\, d\theta$

(c) $\displaystyle\frac{1}{2}\int_{\pi/6}^{5\pi/6} (8\sin^2\theta - 1)\, d\theta$

(d) $\displaystyle\frac{9\pi}{4} - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1+\sin\theta)^2\, d\theta$

(e) None of these.

23. $\displaystyle\lim_{x\to\infty} 2xe^{-x^2}\int_0^x e^{t^2}\, dt =$

(a) 0

(b) 1/2

(c) 1

(d) e

(e) The limit does not exist.

24. If $\dfrac{dy}{dx} = y\tan x$ and $y = 3$ when $x = 0$, then, when $x = \pi/3$, $y =$

(a) $\ln\sqrt{3}$

(b) $\ln 3$

(c) $\dfrac{3}{2}$

(d) $\dfrac{3\sqrt{3}}{2}$

(e) 6

25. If 100 grams of a radioactive substance decays to 50 grams in two years, then, to the nearest gram, the amount left after 3 years is:

(a) 28

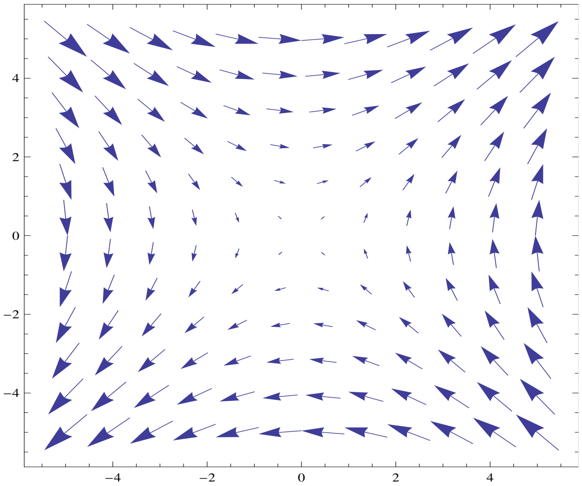
(b) 32

(c) 35

(d) 38

(e) 40

26. Which differential equation has the slope field



(a) $y' = x$
(b) $y' = \dfrac{x}{y}$
(c) $y' = xy$
(d) $y' = \dfrac{y}{x}$
(e) $y' = x^2 y$

27. Which of the following series converges to 2?

I. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ II. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ III. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^n}$

(a) I only
(b) II only
(c) III only
(d) I and III
(e) II and III

28. Let f be a function such that $|f^{(n)}(x)| \leq 2$ for all x and n. Find the least integer n such that the Taylor polynomial of degree n at $x = 0$ approximates $f(1/2)$ to within 0.0001.

(a) 2
(b) 3
(c) 4
(d) 5
(e) 6

29. The function f is infinitely differentiable, $f(2) = 4$, and

$$f^{(n)}(2) = \frac{(n-1)!}{3^n} \quad \text{for all } n \geq 1.$$

The interval of convergence of the Taylor series for f in powers of $x - 2$ is:

(a) $-1 \leq x < 5$
(b) $-3 < x < 3$
(c) $-3 \leq x \leq 3$
(d) $-1 < x < 5$
(e) $-1 < x \leq 5$

30. Suppose that the power series $\sum_{k=0}^{\infty} a_k(x-1)^k$ converges at $x = 3$. Which of the following series must be convergent?

I. $\sum_{k=0}^{\infty} a_k$

II. $\sum_{k=0}^{\infty} a_k 3^k$

III. $\sum_{k=0}^{\infty} (-1)^k a_k$

IV. $\sum_{k=0}^{\infty} (-1)^k a_k 2^k$

(a) I only

(b) II and III

(c) I and III

(d) II, III, and IV

(e) III and IV