University of Houston
Mathematics Contest, Spring 2013 Calculus Test

NAME: ____

SCHOOL: ____

1. $\lim\limits_{x\to 0} \sec\left(\dfrac{2\sin 3\pi x}{9x}\right) =$

   (a) 2
   (b) $-2/\sqrt{3}$
   (c) $-2/\sqrt{2}$
   (d) -2
   (e) The limit does not exist

2. Let f be some function for which you know only that

   $$\text{if} \qquad 0 < |x-4| < 1, \qquad \text{then} \qquad |f(x)-4| < 0.1.$$

   Which of the following statements are necessarily true?

   I. If $|x-4.5| < 0.3$, then $|f(x)-4| < 0.1$.

   II. If $|x-4| < 0.1$, then $|f(x)-4| < 0.01$.

   III. If $0 < |x-4| < 0.5$, then $|f(x)-4| < 0.1$.

   IV. $\lim\limits_{x\to 3} f(x) = 4$.

   (a) II and IV
   (b) III only
   (c) I and III
   (d) I and II
   (e) II, III and IV

3. What is $\lim\limits_{h\to 0} \dfrac{\cos(3x+6h)-\cos 3x}{h}$?

   (a) $6\sin 3x$
   (b) $-6\sin 3x$
   (c) $3\cos 3x$
   (d) $-3\sin 3x$
   (e) $6\cos 3x$

4. A function f is defined on an interval $[a, b]$. Which of the following statements could be false?

   I. If f is differentiable on (a, b) and if f has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

   II. If f is continuous on $[a, b]$, and if $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   III. If f is continuous on $[a, b]$ and there is a point c in (a, b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

   IV. If f is differentiable on an interval $I \supset [a, b]$, and if $f(a)$ and $f(b)$, have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

   (a) II only
   (b) II and IV
   (c) I, III and IV
   (d) II and III
   (e) I and III

5. Set

$$g(x) = \begin{cases} x^3 - 4 & 0 < x < 2, \\ ax^2 + bx & 2 \leq x < 5. \end{cases}$$

   The values of a and b such that g is differentiable on $(0, 5)$ are:

   (a) $a = 5,\ b = -8$
   (b) $a = 5/3,\ b = -8/3$
   (c) $a = -3,\ b = 8$
   (d) $a = 3/2,\ b = -1$
   (e) $a = -2,\ b = 6$

6. The curve $x^3 + x \tan y = 27$ passes through $(3, 0)$. Use local linearization to estimate the value of y at 3.1.

   (a) -2.7
   (b) -0.9
   (c) 0.6
   (d) 0.1
   (e) -2.1

7. The line normal to $3x^2 + 2x + 4y + y^2 = 3$ at the point where $x = m$ is parallel to the y-axis. What is m?

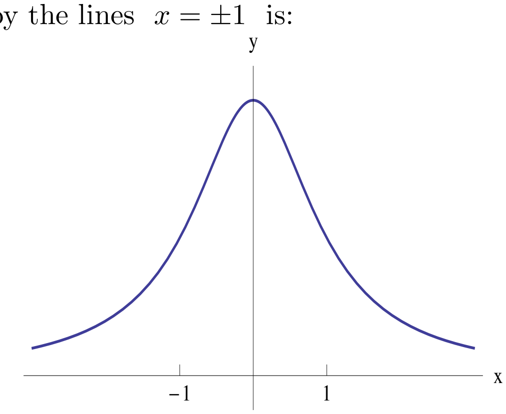
(a) $2/3$
(b) -2
(c) $-1/3$
(d) -3
(e) $1/3$

8. Suppose that f is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Suppose also that $f(0) = 5$ and $f(4) = -3$. Which of the following statements is not necessarily true?

(a) The Mean-Value Theorem applies to f.
(b) f is integrable on $[0, 4]$.
(c) There exists a number $c \in (0, 4)$ such that $f'(c) = -1$.
(d) There exists a number $c \in (0, 4)$ such that $f(c) = \pi$.
(e) If c is any number such that $0 < c < 4$, then $\lim_{x \to c} f(x)$ exists.

9. The graph of $f(x) = \dfrac{4}{1 + x^2}$ is shown below. The area of the region bounded above by the line $y = 4$, below by the curve, and on the sides by the lines $x = \pm 1$ is:

(a) $4 - \pi/4$
(b) $8 - 2\pi$
(c) $8 - \pi$
(d) $8 - \pi/2$
(e) $2\pi - 4$



10. The position of a particle moving along the x-axis is given by

$$s(t) = t^3 - 12t^2 + 45t + 4$$

for $t \geq 0$. For what values of t is the speed of the particle increasing?

(a) $3 < t < 4$ only
(b) $t > 4$ only
(c) $t > 5$ only
(d) $0 < t < 3$ and $t > 5$
(e) $3 < t < 4$ and $t > 5$

11. The maximum value of the function $f(x) = x^2 e^{-x}$ is:

(a) $4/e^2$
(b) $2/e$
(c) $4e^2$
(d) $1/e$
(e) $4e$

12. The base of a solid is the region in the xy-plane bounded by $x^2 = 4y$ and the line $y = 2$. Each plane section of the solid perpendicular to the y-axis is a square. The volume of the solid is:

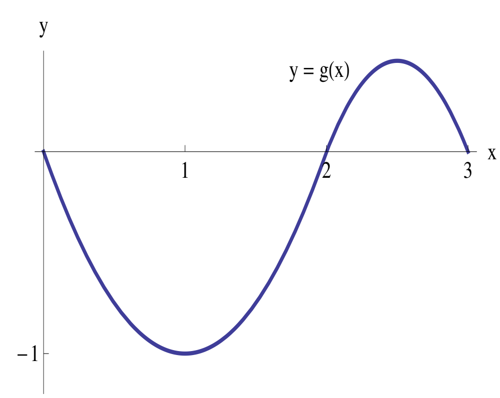
(a) 16
(b) 24
(c) 28
(d) 32
(e) 48

13. The region in the first quadrant bounded by the graph of $y = e^{-x^2}$, the vertical line $x = 1$, and the x-axis is revolved about the y-axis. Find the volume of the solid that is generated.

(a) π/e
(b) $\pi(e-1)$
(c) $\pi - \pi/e$
(d) $\pi - e$
(e) $\pi + \pi/e$

14. $g(x) = \int_0^x f(x)\,dx$. The graph of g is shown below. Which of the following must be true?

I. $\int_0^3 f(x)\,dx = 0$ II. $\int_1^2 f(x)\,dx = 1$ III. $\int_3^2 f(x)\,dx = 0$



(a) II and III only
(b) II only
(c) I and III only
(d) I and II only
(e) I, II and III

15. If f is a continuous function and $F(x) = \int_0^x \left[(t^2+2) \int_t^3 f(u)\, du \right] dt$, then $F''(3) =$

(a) $-6f(3)$
(b) $-11f(3)$
(c) $11f'(3)$
(d) $9f'(3)$
(e) $11f(3)$

16. A curve in the plane is defined by the parametric equations: $x = e^{2t} + 2e^{-t}$, $y = e^{2t} - 3e^{t}$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $2x - 7y = 24$
(b) $5x - 6y = -11$
(c) $7x + 2y = 12$
(d) $2x + 7y = 18$
(e) $5x - 8y = 10$

17. The function $F(x) = 3 + \int_{x^2}^{4} \sqrt{4+3t}\, dt$ has an inverse. $\left(F^{-1}\right)'(3) =$

(a) $1/4$
(b) $-1/4$
(c) $1/8$
(d) $-1/16$
(e) $1/16$

18. The x-coordinate of point (x, y) moving along the curve $y = x^2 + 1$ is increasing at the constant rate of $3/2$ units per second. The rate, in units per second, at which the distance from the origin is changing at the instant the point has coordinates $(1, 2)$ is:

(a) $\dfrac{7\sqrt{5}}{10}$
(b) $\dfrac{3\sqrt{5}}{2}$
(c) $\dfrac{4\sqrt{5}}{5}$
(d) $\dfrac{3\sqrt{5}}{5}$
(e) $\dfrac{5\sqrt{5}}{2}$

19. A curve in the plane is defined by the parametric equations: $x = \frac{1}{3}t^3 - t + 2,\ y = t^2 + 4,\quad t \in [1, 3]$. Find the length of the curve.

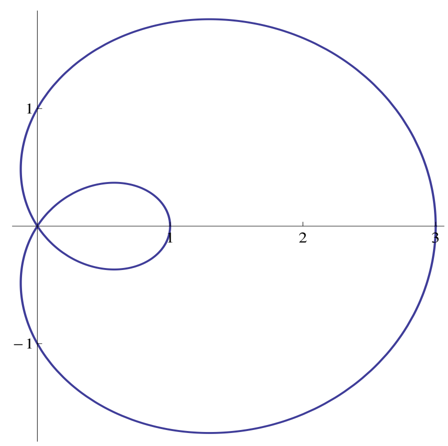
(a) 40/3
(b) 26/3
(c) 32/3
(d) 14/3
(e) 34/3

20. Find k if the average value $f(x) = x^3 + 1$ on $[0, k]$ is 17.

(a) $\sqrt[3]{51}$
(b) $\sqrt[4]{68}$
(c) 3
(d) 4
(e) $\sqrt{48}$

21. Which of the following represents the area enclosed by the inner loop of the graph of $r = 1 + 2\cos\theta$?

(a) $\dfrac{1}{2}\displaystyle\int_{5\pi/6}^{7\pi/6} (1 + 2\cos\theta)^2\, d\theta$
(b) $\dfrac{1}{2}\displaystyle\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2\, d\theta$
(c) $\dfrac{1}{2}\displaystyle\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta$
(d) $\dfrac{1}{2}\displaystyle\int_{7\pi/6}^{11\pi/6} (1 + 2\cos\theta)^2\, d\theta$
(e) $\dfrac{1}{2}\displaystyle\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta$



22. Find a such that $\displaystyle\lim_{x\to 0} \frac{e^{ax^2} - \cos 8x}{x^2} = 64$

(a) 48
(b) 32
(c) 28
(d) 16
(e) No value of a exists.

23. If $\dfrac{dy}{dx} = y \cot x$ and $y = 4$ when $x = \pi/2$, then, when $x = 2\pi/3$, $y =$

(a) $2\sqrt{3}$

(b) $2 \ln \sqrt{3}$

(c) $4 \ln(\sqrt{3}/2)$

(d) 2

(e) $2 \ln \sqrt{3}$

(f) $4\sqrt{3}$

24. The rate at which a certain bacteria population grows is proportional to number of bacteria present. Initially there were 1,000 bacteria present and the population doubled in 6 hours. Approximately how many hours will it take for the population to reach 10,000?

(a) 17.4

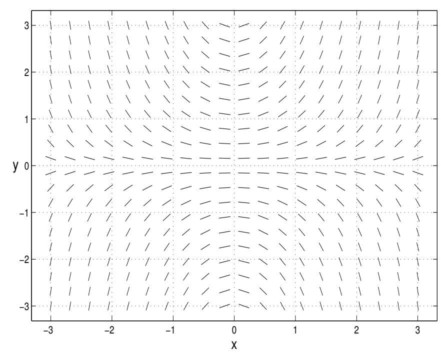
(b) 31.2

(c) 14.5

(d) 19.9

(e) 24.7

25. Which differential equation has the slope field



(a) $\dfrac{dy}{dx} = xy$

(b) $\dfrac{dy}{dx} = \dfrac{y}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(d) $\dfrac{dy}{dx} = x^2 y$

(e) $\frac{dy}{dx} = x + y$

26. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are necessarily true?

I. If $a_n > 1$ for all n and $a_n \to L$, then $L > 1$.

II. If $\{a_n\}$ is not bounded below, then it diverges.

III. If a_n converges, then it is bounded above.

IV. If $\{a_n\}$ is bounded, then it converges.

(a) III only

(b) I, III

(c) II, IV

(d) II, III, IV

(e) II, III

27. Which of the following series converges to 2?

I. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ III. $\sum_{n=0}^{\infty} \frac{3(-1)^n}{2^n}$

(a) I only
(b) II only
(c) III only
(d) I and III
(e) II and III

28. A function f is infinitely differentiable and has the property that $\left|f^{(k)}(x)\right| \leq k2^k$ for all $x \in (-1, 1)$ and all k. Find the least integer n such that the Taylor polynomial of degree n in powers of x for f approximates $f(1/4)$ to within 0.0005

(a) 3
(b) 4
(c) 5
(d) 6
(e) 7

29. The function f is infinitely differentiable, $f(1) = 3$, and

$$f^{(n)}(1) = \frac{(n-1)!}{2^n} \quad \text{for all } n \geq 1.$$

The interval of convergence of the Taylor series for f in powers of $x - 1$ is:

(a) $-1 \leq x < 3$
(b) $0 < x \leq 2$
(c) $-2 \leq x \leq 2$
(d) $-1 < x < 2$
(e) $-1 < x \leq 3$

30. Suppose that the power series $\sum_{k=0}^{\infty} a_k(x-2)^k$ converges at $x = 4$. Which of the following series must be convergent?

I. $\sum_{k=0}^{\infty} a_k 3^k$ II. $\sum_{k=0}^{\infty} a_k$ III. $\sum_{k=0}^{\infty} (-1)^k a_k$ IV. $\sum_{k=0}^{\infty} (-1)^k a_k 2^k$

(a) II only
(b) I and III
(c) II and III
(d) II, III, and IV
(e) III and IV