UNIVERSITY OF HOUSTON
HIGH SCHOOL MATHEMATICS CONTEST

Spring 2017 Calculus Test

NAME: ____

SCHOOL: ____

1. $\lim\limits_{x\to 0} \tan\left(\dfrac{5\sin 2\pi x}{6x}\right) =$

   (a) -1

   (b) $\sqrt{3}$

   (c) $-1/\sqrt{3}$

   (d) $-\sqrt{3}$

   (e) The limit does not exist

2. Let f and g be differentiable functions which satisfy the following conditions:

x:	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	-1	3	5	0
1	-1	2	-1	2

   If $h(x) = \dfrac{f(g(x))}{x^2}$, then $h'(1) =$

   (a) -2

   (b) 5

   (c) 8

   (d) -4

   (e) 4

3. $\lim\limits_{h\to 0} \dfrac{1}{h}\ln\left(\dfrac{3+h}{3}\right) =$

   (a) The limit does not exist.

   (b) e^3

   (c) 1

   (d) $\ln 3$

   (e) $1/3$

4. A function f is defined on an interval $[a, b]$. Which of the following statements could be false?

I. If f is differentiable on (a, b), and if $f(a)$ and $f(b)$, have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

II. If f is continuous on $[a, b]$, and if $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

III. If f is continuous on $[a, b]$ and there is a point c in (a, b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

IV. If f is differentiable on (a, b) and if f has no zeros on $[a, b]$, then $f(a)$ and $f(b)$ have the same sign.

(a) IV only
(b) II, III
(c) I, III, IV
(d) II, IV
(e) I, III

5. Set

$$g(x) = \begin{cases} a\sqrt{x+2} & 0 < x < 2, \\ bx - 4 & 2 \leq x < 4. \end{cases}$$

The values of a and b such that g is differentiable on $(0, 4)$ are:

(a) $a = -8/3,\ b = -2/3$
(b) $a = 2,\ b = 4$
(c) $a = 12/3,\ b = 2/3$
(d) $a = -8,\ b = -2$
(e) $a = -4,\ b = -2$

6. When the local linearization of $f(x) = \sqrt{4 + \sin x}$ at $x = 0$ is used, an estimate for $f(0.12)$ is:

(a) 2.00
(b) 2.03
(c) 2.06
(d) 2.12
(e) 2.16

7. The normal line to the curve $2x^3 + 2y^2 = 5xy$ at the point $(1, 2)$ has y-intercept:

(a) $(0, 11/4)$
(b) $(0, -5/4)$
(c) $(0, -1/2)$
(d) $(0, 3/2)$
(e) $(0/1/3)$

8. The function $f(x) = 3\sqrt{x} - 4x,\ 1 \le x \le 4$ satisfies the conditions of the Mean-Value Theorem for Derivatives. The values of c that satisfy the conclusion of the theorem are:

(a) $c = \sqrt{3/2}$
(b) $c = 3/2$
(c) $c = 9/4$
(d) $c = 2/3$
(e) $c = \sqrt{2/3}$

9. The average value of $f(x) = \sec^2 x$ on the interval $(\pi/6, \pi/4)$ is:

(a) $8/\pi$
(b) $\dfrac{12\sqrt{3} - 12}{\pi}$
(c) $\dfrac{6\sqrt{2} - 6}{\pi}$
(d) $\dfrac{12 - 4\sqrt{3}}{\pi}$
(e) $\dfrac{6 - 6\sqrt{2}}{\pi}$

10. A particle is moving along the x-axis so that its position at time $t,\ 0 \le t \le 10$ is

$$s(t) = \int_0^t \ln\left(u^2 - 3u + 3\right)\, du.$$

During which time intervals is the particle moving to the left?

(a) $2 < t \le 10$
(b) $1 < t < 2$
(c) The particle never moves left.
(d) $0 \le t < 1$
(e) $0 < t < 5$

11. The maximum value of the function $f(x) = \dfrac{\ln x}{x^2}$ is

(a) $2e$
(b) $2/\sqrt{e}$
(c) $\sqrt{e}$
(d) $2/e$
(e) $1/2e$

12. The region bounded by the graph of $y = \cos x$, $-\pi/2 \le x \le \pi/2$, is divided into two regions by the line $x = k$. If the area of the region for $-\pi/2 \le x \le k$ is three times the area of the region for $k \le x \le \pi/2$, the $k =$

(a) $\arcsin(1/4)$
(b) $\pi/6$
(c) $\arcsin(1/3)$
(d) $\pi/4$
(e) $\pi/3$

13. Let R be the region bounded by the graph of $f(x) = \sqrt{x+1}$, the vertical line $x = 8$, and the x-axis. Find the volume of the solid generated when R is revolved about the line $y = 3$.

(a) $\dfrac{135\pi}{2}$
(b) $\dfrac{99\pi}{2}$
(c) $\dfrac{189\pi}{2}$
(d) $\dfrac{119\pi}{2}$
(e) $\dfrac{137\pi}{2}$

14. The region in the first quadrant bounded by the graph of $y = \sin x$, $0 \le x \le \pi$, is revolved about the y-axis. Find the volume of the solid that is generated.

(a) π^2
(b) $4\pi/2$
(c) $2\pi^2$
(d) π
(e) 2π

15. If f is a continuous function and $F(x) = \int_{x^2}^{4} (x^2 - x) f(u)\, du$, then $F'(2) =$

(a) $4f(4)$
(b) $-8f(2)$
(c) $-12f(2)$
(d) $3f(4)$
(e) $-8f(4)$

16. A curve in the plane is defined by the parametric equations: $x = e^{2t} - 1,\ y = e^t + 2e^{2t}$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $7x - 4y = 11$
(b) $3x + 8y = -12$
(c) $9x - 4y = -13$
(d) $18x - 4y = 21$
(e) $4x + 9y = -14$

17. The function $f(x) = 4 + \int_4^{x^2} \sqrt{9 + t^2}\, dt$ has an inverse. Calculate $\left(f^{-1}\right)'(4)$.

(a) $1/5$
(b) 5
(c) 20
(d) $1/20$
(e) $1/50$

18. The x-coordinate of point (x, y) moving along the curve $y = x^2 + 1$ is increasing at the constant rate of $3/2$ units per second. The rate, in units per second, at which the distance from the origin is changing at the instant the point has coordinates $(1, 2)$ is:

(a) $\frac{7\sqrt{5}}{10}$
(b) $\frac{3\sqrt{5}}{2}$
(c) $\frac{4\sqrt{5}}{5}$
(d) $\frac{3\sqrt{5}}{5}$
(e) $\frac{5\sqrt{5}}{2}$

19. A curve in the plane is defined by the parametric equations:

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t \in [0, \pi].$$

Find the length of the curve.

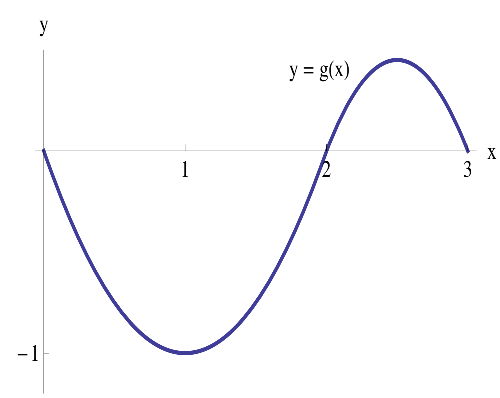
(a) $\frac{1}{2}\pi^2$
(b) π
(c) $\frac{1}{4}\pi$
(d) π^2
(e) $\frac{1}{4}\pi^2$

20. A ball rebounds to two-thirds of the height from which it falls. If it is dropped from a height of 6 feet and is allowed to continue bouncing indefinitely, what is the total distance it travels?

(a) 32 feet
(b) 52/3 feet
(c) 36 feet
(d) 30 feet
(e) 64/3 feet

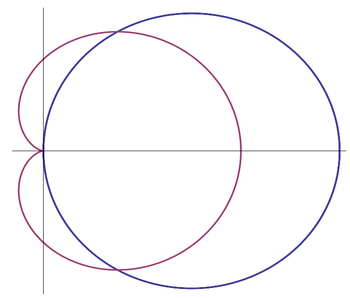
21. $g(x) = \int_0^x f(x)\,dx$. The graph of g is shown below. Which of the following must be true?

$$\text{I. } \int_0^3 f(x)\,dx = 0 \qquad \text{II. } \int_1^2 f(x)\,dx = 1 \qquad \text{III. } \int_3^2 f(x)\,dx = 0$$



(a) II and III only
(b) II only
(c) I and III only
(d) I and II only
(e) I, II and III

22. The area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$ is given by



(a) $\int_{-\pi/6}^{\pi/6} (2\cos\theta - 1)^2\, d\theta$

(b) $\int_{0}^{\pi/3} \left[8\cos^2\theta - 2\cos\theta - 1\right]\, d\theta$

(c) $\frac{1}{2}\int_{-\pi/3}^{\pi/3} (8\cos^2\theta - 1)\, d\theta$

(d) $\frac{9\pi}{4} - \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1 + \cos\theta)^2\, d\theta$

(e) $\int_{0}^{\pi/6} \left[9\cos^2\theta - (1+\cos\theta)^2\right]\, d\theta$

23. $\lim_{n\to\infty} (1 + e^n)^{2/n} =$

(a) 0

(b) e^{-1}

(c) e^2

(d) e^{-2}

(e) 1

24. If $\frac{dy}{dx} = y\tan x$ and $y = 3$ when $x = 0$, then, when $x = \pi/3$, $y =$

(a) $\ln\sqrt{3}$

(b) $\ln 3$

(c) $\frac{3}{2}$

(d) $\frac{3\sqrt{3}}{2}$

(e) 6

25. At 12 noon on Jan. 1, the count in a bacteria culture was 400; at 4:00 pm the count was 1200. Let $P(t)$ denote the bacteria count at time t and assume that the culture obeys the population growth law. The bacteria count at 10 am on Jan. 1 was (approximately)?

(a) 231

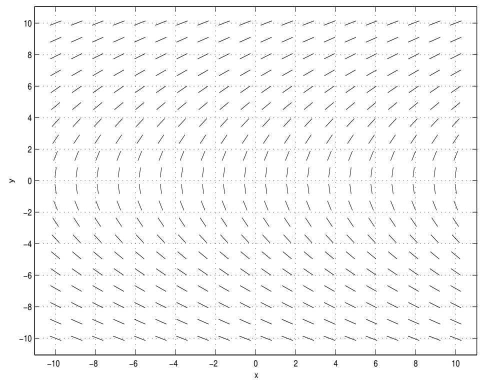
(b) 198

(c) 259

(d) 214

(e) 271.

26. Which differential equation has the slope field



(a) $y' = x$

(b) $y' = \dfrac{x}{y}$

(c) $y' = xy$

(d) $y' = \dfrac{y}{x}$

(e) $y' = x^2 y$

27. Which of the following series converges to 2?

I. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ II. $\displaystyle 5 - \sum_{n=1}^{\infty} 2\left(\frac{3}{5}\right)^n$ III. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^n}$

(a) I only

(b) II only

(c) III only

(d) I and III

(e) I, II and III

28. Let f be a function such that $|f^{(n)}(x)| \leq 2$ for all x and n. Find the least integer n such that the Taylor polynomial of degree n at $x = 0$ approximates $f(1/2)$ to within 0.0001.

(a) 2

(b) 3

(c) 4

(d) 5

(e) 6

29. The function f is infinitely differentiable, $f(1) = 4$, and

$$f^{(n)}(1) = \frac{(n-1)!}{2^n} \quad \text{for all } n \geq 1.$$

The interval of convergence of the Taylor series for f in powers of $x - 1$ is:

(a) $-1 \leq x < 3$

(b) $-1 < x < 4$

(c) $-1 \leq x < 1$

(d) $0 < x \leq 2$

(e) $-1 < x \leq 3$

30. The function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2} & x \neq 0, \\ \frac{1}{2} & x = 0. \end{cases}$$

has a Taylor series expansion in powers of x. $f^{(10)}(0) = ?$

(a) $\dfrac{1}{12!}$

(b) $\dfrac{1}{10!}$

(c) $\dfrac{-1}{12 \cdot 11}$

(d) $\dfrac{-1}{10 \cdot 9}$

(e) $\dfrac{1}{11 \cdot 10}$