Project Problem
2017 Math Contest
University of Houston

Ellipse-Or-No

Directions

Rules: Teams can consist of no more than 4 students who are zoned to the same school, or a feeder pair of schools (middle school and high school).

Directions: Assemble your team. Then write a written report that introduces your team, details their contributions to the project, and discusses your team's solutions to the problems. Also, create an associated video presentation (no longer than 10 minutes) that discusses Ellipses. Make it fun. Everyone on your team must play a role in the video, and credits must appear at the end of the video, detailing the contribution of each team member.

Project solutions can be submitted by sending an email to jjmorgan@central.uh.edu by 9am on January 28th. Your email must include the name of the school, the name(s) of the team members, and link(s) to your solution file(s). Do NOT send your solution as an attachment.

Please use the subject line TEAM PROJECT.

You can email questions and comments, prior to your submission, to jjmorgan@central.uh.edu. As above, use the subject line TEAM PROJECT.

Project Evaluation:

Project submissions will be evaluated using the following criteria:

- Organization.
- Creativity.
- Clarity of presentation.
- Correctness.

Introduction

We refer to 2 dimensional space as R^2, and give the definition

$$R^2 = \{(x, y) | x \text{ and } y \text{ are real numbers}\}$$

You might recognize this as the xy plane. We define addition, subtraction and scalar multiplication on R^2 via

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) - (c, d) = (a - c, b - d)$$
$$\beta(a, b) = (\beta a, \beta b)$$

whenever $\beta,\ a,\ b,\ c$ and d are real numbers. Finally, the distance between a point (a, b) and the origin $(0,0)$ in R^2 is given by

$$|(a, b)| = \sqrt{a^2 + b^2}$$

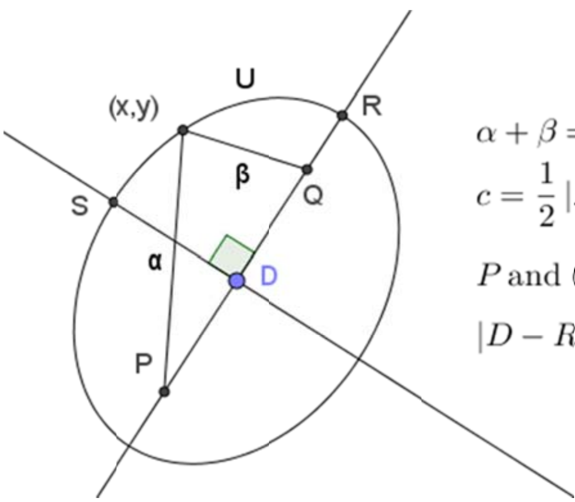
and the distance between two points (a, b) and (c, d) in R^2 is given by $|(a, b) - (c, d)|$.

Definition: We say that a subset U of R^2 is an ellipse if and only if there are points P and Q in R^2, and a scalar $L > |P - Q|$, so that U is the set of points (x, y) in R^2 satisfying

$$|(x, y) - P| + |(x, y) - Q| = L$$

The points P and Q in the definition are referred to as the foci of the ellipse, and the center of the ellipse is given by $\frac{1}{2}(P + Q)$ (i.e. the midpoint of the line segment joining P and Q).

If U is an ellipse with associated foci P and Q, and scalar L, and we define $a = \frac{L}{2}$, $c = \frac{1}{2}|P - Q|$ and $b = \sqrt{a^2 - c^2}$, then you may be familiar with the figure below.



$$\alpha + \beta = L$$
$$c = \frac{1}{2}|P - Q|,\ a = \frac{L}{2} \text{ and } b = (a^2 - c^2)^{\frac{1}{2}}$$
P and Q are the foci and $D = \frac{1}{2}(P + Q)$ is the center
$|D - R| = a$ and $|D - S| = b$

In exercises 1 and 2, you are asked to verify properties associated with the points R and S in the figure.

Exercises: It is not necessary to solve all of exercises 1-8. Every team must provide the video requested in exercise 9.

1. Suppose U is an ellipse with associated foci P and Q, and scalar L. Assume P and Q are distinct points, $a = \frac{L}{2}$, $c = \frac{1}{2}|P - Q|$ and $b = \sqrt{a^2 - c^2}$. Let D be the center of the ellipse. Show that the points $D + \frac{a}{c}(Q - D)$ and $D - \frac{a}{c}(Q - D)$ are the only 2 points in U that are a distance a from D, all other points in U are closer to D, and U is symmetric about the line through $D + \frac{a}{c}(Q - D)$ and $D - \frac{a}{c}(Q - D)$. **Note:** The point R in the figure above is $D + \frac{a}{c}(Q - D)$.
2. Suppose U is an ellipse with associated foci P and Q, and scalar L. Assume P and Q are distinct points, $a = \frac{L}{2}$, $c = \frac{1}{2}|P - Q|$ and $b = \sqrt{a^2 - c^2}$. Let D be the center of the ellipse. Let ℓ_1 be the line through $D + \frac{a}{c}(Q - D)$ and $D - \frac{a}{c}(Q - D)$, and ℓ_2 be the line perpendicular to ℓ_1 that passes through D. Show that ℓ_2 intersects U in exactly 2 points, and these points are the only 2 points in U that are a distance b from D. Show that all other points in U are further from D. Finally, show that U is symmetric about the line ℓ_2. **Note:** The point S in the figure above is one of the points of intersection of ℓ_2 with U.
3. Prove that the set U is an ellipse with center $D = (x_0, y_0)$ if and only if there are scalars r, s and t so that $r > 0$, $rs - t^2 > 0$, and U is the set of points (x, y) solving the equation
$$r(x - x_0)^2 + s(y - y_0)^2 + 2t(x - x_0)(y - y_0) = 1$$
4. Ellipse-Or-No? Watch **VIDEO A** and determine whether the object being drawn is an ellipse.
5. Ellipse-Or-No? Watch **VIDEO B** and determine whether the object being drawn is an ellipse.
6. Ellipse-Or-No? Watch **VIDEO C** and determine whether the object being drawn is an ellipse.
7. Prove that if U is an ellipse, and P, Q, R, S and T are distinct points in U, then U is the only ellipse containing the points P, Q, R, S and T.
8. Ellipse-Or-No? Watch **VIDEO D** and determine whether the object being drawn is an ellipse.
9. Create a video presentation (no longer than 10 minutes) that discusses Ellipses. Make it fun. Everyone on your team must play a role in the video, and credits must appear at the end of the video, detailing the contribution of each team member.