UNIVERSITY OF HOUSTON
HIGH SCHOOL MATHEMATICS CONTEST

Spring 2018 Calculus Test

NAME: ______________________________

SCHOOL: ______________________________

1. Let f be some function for which you know only that

$$\text{if} \qquad 0 < |x-3| < 1, \qquad \text{then} \qquad |f(x)-5| < 0.1.$$

Which of the following statements are necessarily true?

I. If $|x-3| < 0.1$, then $|f(x)-5| < 0.01$.

II. If $0 < |x-3| < 0.5$, then $|f(x)-5| < 0.1$.

III. If $|x-3.4| < 0.4$, then $|f(x)-5| < 0.1$.

IV. $\lim_{x\to 3} f(x) = 5$.

(a) II and IV
(b) III only
(c) II and III
(d) I and II
(e) II, III and IV

2. $\lim_{x\to 0} \tan\left(\dfrac{2\sin 2\pi x}{3x}\right) =$

(a) 1
(b) $-\sqrt{3}$
(c) $1/\sqrt{3}$
(d) $\sqrt{3}$
(e) $-1/\sqrt{3}$

3. What is $\lim_{h\to 0} \dfrac{\cos(4+3h)-\cos 4}{2h}$?

(a) $-\frac{3}{2}\sin 4$
(b) $\frac{1}{2}\sin 4$
(c) $-\frac{1}{2}\sin 4$
(d) $\frac{3}{2}\cos 4$
(e) $-\sin 4$

4. If $\dfrac{d}{dx} f(x) = g(x)$ and if $h(x) = e^{-2x}$, then $\dfrac{d}{dx} f(h(x)) =$

(a) $-2g\left(e^{-2x}\right)$

(b) $-2e^{-2x}\, g(x)$

(c) $e^{-2x} g'(x)$

(d) $-2e^{-2x}\, g\left(e^{-2x}\right)$

(e) $e^{-2x} g\left(e^{-2x}\right)$

5. The values of A and B such that

$$f(x) = \begin{cases} Ax^3 + Bx + 2, & x \leq 2 \\ Bx^2 - A, & x > 2 \end{cases}$$

is everywhere differentiable are:

(a) $A = -2/3,\ B = -8/3$

(b) $A = -2/5,\ B = -6/5$

(c) $A = -8,\ B = -2$

(d) $A = 2/15,\ B = 8/5$

(e) $A = -2,\ B = -8$

6. Set $f(x) = \dfrac{4}{1+x^2}$, and let $H(x) = \displaystyle\int_0^x f(t)\, dt$. The local linearization of H at $x = 1$ is

(a) $y = \pi + 2x$

(b) $y = 2x + \pi - 2$

(c) $y = 2x + \pi - 1$

(d) $y = \pi - 2x - 2$

(e) $y = -2x + 2\ln 2$

7. If $f'(x) = 6(x-3)(x-9)^2$, which of the following is true about f?

(a) f has a local maximum at $x = 3$ and a local minimum at $x = 9$.

(b) f has a point of inflection at $x = 3$ and a local maximum at $x = 9$.

(c) f has a local minimum at $x = 3$ and a local maximum at $x = 9$.

(d) f has a point of inflection at $x = 3$ and a local minimum at $x = 9$.

(e) f has a local minimum at $x = 3$ and a point of inflection at $x = 9$.

8. The line normal to $3x^2 - 2x + 4y + y^2 = 3$ at the point where $x = m$ is parallel to the y-axis. What is m?

(a) $1/3$
(b) $1/6$
(c) $-1/3$
(d) 3
(e) -2

9. The value of c that satisfies the Mean Value Theorem for Derivatives on the interval $[0, 5]$ for the function $f(x) = x^3 - 6x + 2$ is:

(a) $\sqrt{26/3}$
(b) $2\sqrt{2}$
(c) $5\sqrt{3}/3$
(d) $\sqrt{37/3}$
(e) $4\sqrt{3}/3$

10. A rectangle with one side on the x-axis is inscribed in the triangle formed by the lines $y = 2x$, $y = 0$, and $2x + y = 12$. In square units, the maximum area of such a rectangle is:

(a) $16/3$
(b) 9
(c) $10/3$
(d) 6
(e) 7

11. The position of a particle moving along the x-axis is given by

$$s(t) = t^3 - 9t^2 + 15t + 5$$

for $t \geq 0$. For what values of t is the speed of the particle increasing?

(a) $1 < t < 3$ only
(b) $t > 3$ only
(c) $1 < t < 3$ and $t > 5$
(d) $0 < t < 1$ and $t > 5$
(e) $t > 5$ only

12. The maximum value of the function $f(x) = \dfrac{\ln(x^2)}{x^2}$ is:

(a) $2/e^2$
(b) $\sqrt{e}$
(c) $4e^2$
(d) $1/e$
(e) $2/\sqrt{e}$

13. The base of a solid is the region in the xy-plane bounded by $x^2 = 4y$ and the line $y = 2$. Each plane section of the solid perpendicular to the y-axis is an equilateral triangle. The volume of the solid in cubic units is:

(a) $4\sqrt{3}$
(b) $16\sqrt{3}$
(c) 16
(d) $8\sqrt{3}$
(e) 12

14. The region in the first quadrant bounded by $y = \tan x$, $y = 0$, and $x = \frac{\pi}{4}$ is rotated about the x-axis. In cubic units, the volume of the generated solid is:

(a) $\pi - \dfrac{\pi^2}{4}$
(b) $\pi\left(\sqrt{2} - 1\right)$
(c) $\dfrac{3\pi}{4}$
(d) $\pi\left(1 + \dfrac{\pi}{4}\right)$
(e) $1 - \dfrac{\pi}{4}$

15. If $\displaystyle\int_0^4 f(x)\,dx = 5$, $\displaystyle\int_2^4 f(x)\,dx = 7$, and $\displaystyle\int_0^7 f(x)\,dx = 10$, then $\displaystyle\int_7^2 f(x)\,dx =$

(a) 12
(b) 8
(c) -12
(d) -2
(e) -8

16. If f is a continuous function and $F(x) = \int_0^x \left[(t^2+3t) \int_t^2 f(u)\,du \right] dt$, then $F''(2) =$

(a) $7f(2) - 10f'(2)$
(b) $10f(2)$
(c) $11f'(2)$
(d) $7f(2)$
(e) $-10f(2)$

17. A curve in the plane is defined by the parametric equations: $x = e^{2t} + 2e^{-t}$, $y = e^{2t} - 3e^{t}$. An equation for the line tangent to the curve at the point where $t = \ln 2$ is:

(a) $2x - 7y = 24$
(b) $5x - 6y = -11$
(c) $7x + 2y = 12$
(d) $2x + 7y = 18$
(e) $5x - 8y = 10$

18. The function $F(x) = 2 + \int_9^{x^2} \sqrt{9+3t}\,dt$ is differentiable and has an inverse. $\left(F^{-1}\right)'(2) =$

(a) $1/6$
(b) $1/36$
(c) 6
(d) 36
(e) $1/12$

19. The x-coordinate of point (x, y) moving along the curve $y = x^2 - 5$ is increasing at the constant rate of $5/2$ units per second. The rate, in units per second, at which the distance from the origin is changing at the instant the point has x-coordinate 3 is:

(a) 12
(b) 17.5
(c) 13.5
(d) 15
(e) 11.5

20. A curve in the plane is defined by the parametric equations: $x = 3t^2+3,\ y = \frac{2}{3}t^3-2,\ t \in [0,4]$. Find the length of the curve.

(a) 244/3
(b) 98
(c) 196/3
(d) 250/3
(e) 196

21. Find k if the average value of $f(x) = x^2 - 2x$ on $[0,k]$ is 18.

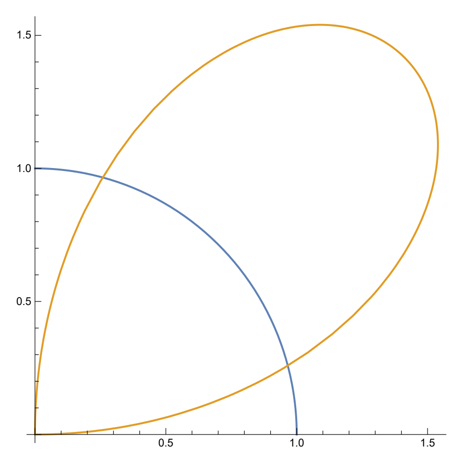
(a) 9
(b) 12
(c) 25/3
(d) 10
(e) 28/3

22. $\displaystyle\int_{3/\pi}^{\infty} \frac{\sin(1/t)}{t^2}\,dt =$

(a) 2
(b) $\sqrt{3}/2$
(c) $1-\sqrt{3}/2$
(d) 1/2
(e) 3/2

23. Find the area of the region which lies inside one loop of the curve $r = 2\sin 2\theta$ and outside the circle $r = 1$. (See the figure.)

(a) $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{6}$
(b) $\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{4}$
(c) $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$
(d) $\dfrac{\pi}{8} + \dfrac{\sqrt{3}}{4}$
(e) $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{6}$



24. A ball rebounds to three-fourths of the height from which it falls. If it is dropped from a height of 8 feet and is allowed to continue bouncing indefinitely, what is the total distance it travels?

(a) 64 feet
(b) 60 feet
(c) 56 feet
(d) 72 feet
(e) 48 feet

25. Find $\lim_{x\to\infty} (e^{2x}+2)^{1/x}$.

(a) 1
(b) $\sqrt{e}$
(c) 2
(d) e
(e) e^2

26. If $\dfrac{dy}{dx} = y\cot x$ and $y = 3$ when $x = \pi/6$, then, when $x = 4\pi/3$, $y =$

(a) $-6\sqrt{3}$
(b) $-3\sqrt{3}$
(c) $-3\sqrt{2}$
(d) $3\sqrt{3}$
(e) $4\sqrt{3}$

27. Radioactive materials decay at a rate proportional to the amount present. Two years ago a laboratory had 50 grams of a certain radioactive substance. Today the lab has 40 grams of the substance. How many grams will the lab have four years from now.

(a) 25.6
(b) 31.2
(c) 34.5
(d) 19.4
(e) 23.8

28. $\{a_n\}$ is a sequence of real numbers. Which of the following statements are necessarily true?

I. If $a_n > 1$ for all n and $a_n \to L$, then $L > 1$.

II. If $\{a_n\}$ is not bounded below, then it diverges.

III. If a_n converges, then it is bounded above.

IV. If $\{a_n\}$ is bounded, then it converges.

(a) III only
(b) I, III
(c) II, IV
(d) II, III, IV
(e) II, III

29. The function f is infinitely differentiable, $f(2) = 1$, and

$$f^{(n)}(2) = \frac{(-1)^n (n-1)!}{3^n} \quad \text{for all } n \geq 1.$$

The interval of convergence of the Taylor series for f in powers of $x - 2$ is:

(a) $0 \leq x \leq 4$
(b) $-1 \leq x < 5$
(c) $-3 \leq x < 3$
(d) $-1 < x \leq 5$
(e) $-3 < x < 3$

30. Suppose that the power series $\sum_{k=0}^{\infty} a_k (x-2)^k$ converges at $x = 4$. Which of the following series must be convergent?

I. $\sum_{k=0}^{\infty} a_k (-3)^k$ II. $\sum_{k=0}^{\infty} a_k$ III. $\sum_{k=0}^{\infty} (-1)^k a_k$ IV. $\sum_{k=0}^{\infty} (-2)^k a_k$

(a) II only
(b) I and III
(c) II and III
(d) II, III, and IV
(e) III and IV