Calculus Exam - University of Houston Math Contest
January 30, 2021

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1) If we are told that $\lim_{t\to a} \frac{\sqrt{t}-\sqrt{a}}{t-a} = 2$ then the value of a is

a) 4 b) $\frac{1}{16}$ c) 2021 d) $\frac{1}{4}$ e) None of the other answers provided.

2) Suppose $f(x)$ is a differentiable, invertible function whose graph passes through the point $(2,5)$ where its *normal* line has a slope of -8. What is the slope of the *tangent* line to the graph of the inverse function $y = f^{-1}(x)$ at the point $(5,2)$?

a) 8 b) -8 c) $-\frac{1}{8}$ d) $\frac{1}{8}$ e) None of the other answers provided.

3) The graph of the positive function $y = f(x)$ determines a region over the interval $[0,1]$ that encloses 5 units of area (such as the one shown in the Figure 1 below).

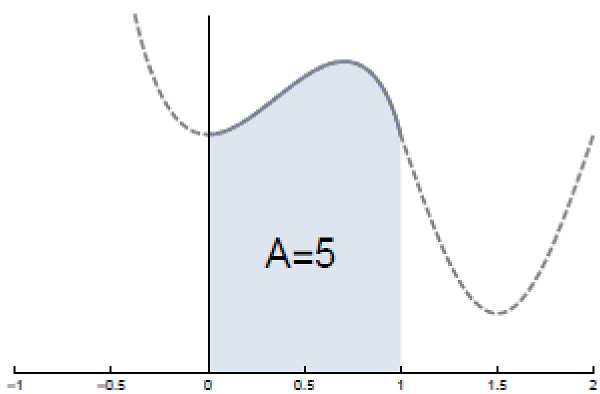


Figure 1

When this region is revolved about the *x*-axis the resulting solid encloses 2021π units of volume. How much volume is enclosed by the solid of revolution obtained by revolving $y = f(x)+1$ about the *x*-axis along the interval $[0,1]$?

a) 2021π units of volume b) 2010π units of volume c) 2032π units of volume
d) 2064π units of volume e) None of the other answers provided.

4) Two particles are moving along the *y*-axis with positions given, respectively, by

$$y_1(t) = \frac{t^4}{3} - \frac{16t^3}{3} + 32t^2 - t + 5 \quad \text{and} \quad y_2(t) = \frac{5t^4}{12} - \frac{20t^3}{3} + 40t^2 + 5t - 1$$

At how many distinct points in time do the two particles share the same acceleration?

a) The accelerations match at three points in time. b) The accelerations match at one point in time.
c) The accelerations match at four points in time. d) The accelerations match at two points in time.
e) None of the other answers provided.

5) For which value of a does the following limit hold? $\lim_{x\to 0}(1+x)^{\frac{a}{x}} = a$

a) $a = 1$ b) $a = 0$ c) $a = \mathrm{e}$ d) $a = 2021$ e) There is no value of a that makes this equation true.

6) At each point (x, y) along a given differentiable curve there is a tangent line with slope $2\cos x + \frac{1}{\pi}$. If the curve passes through the point $(0, 1)$, what is the y-coordinate of the curve when $x = \frac{\pi}{2}$?

a) $y = 0$ b) $y = -\frac{5}{2}$ c) $y = \frac{1}{\pi}$ d) $y = \frac{7}{2}$ e) None of the other answers provided.

7) The differentiable function $F(x)$ is given by $F(x) = \int_0^{1+x+x^2} f(t)\mathrm{d}t$.

The line $y = 2(x+1)+7$ is tangent to the graph of F at the point $(-1, F(-1))$. Based on this information, which of the following statements, if any, are true?

I. The (signed) area between the graph of $y = f(t)$ and the t-axis along the interval $[0, 1]$ equals 7.
II. $f(1) = -2$
III. The average value of f over the interval $[0, 1]$ equals 7.

a) I and II only. b) I, II and III. c) II and III only. d) I and III only.
e) None of the other answers provided.

8) The value of the limit $\lim_{h\to 0}\frac{(x-1+h)^3-(x-1)^3}{h}$ depends on the real number x.
The smallest value of this limit occurs when
a) $x = 1$ b) $x = 8$ c) $x = 0$ d) $x = -1$ e) None of the other answers provided.

Figure 2 is used for Problems 9-11 and is shown below.

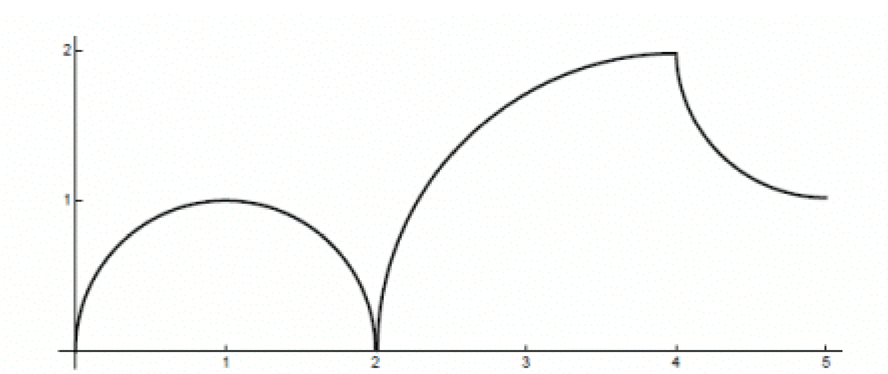


Figure 2: The graph of $y = f(x)$ consists entirely of circular arcs

9) Given the graph of $y = f(x)$ in Figure 2, it follows that $\int_0^2 f(x)\mathrm{d}x - \int_2^4 f(x)\mathrm{d}x + \int_4^5 f(x)\mathrm{d}x =$

a) $2 - \frac{3\pi}{4}$ b) $\frac{5\pi}{2} - 1$ c) $2 + \frac{3\pi}{4}$ d) $\frac{5\pi}{2} + 1$ e) None of the other answers provided.

Figure 2 is shown again for use in Problems 10 and 11.

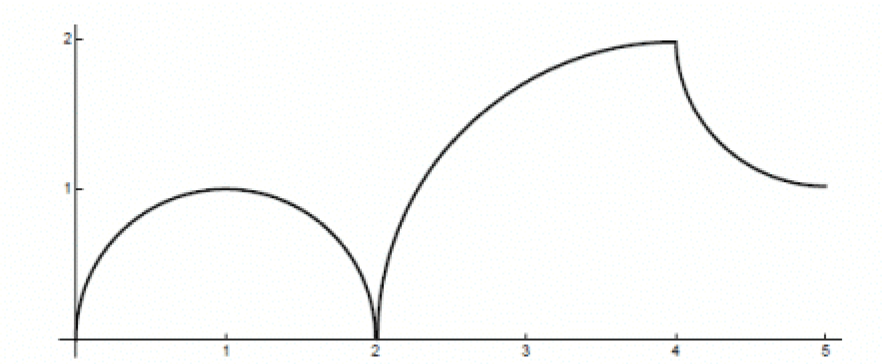


Figure 2: The graph of $y = f(x)$ consists entirely of circular arcs

10) Let n denote the number of points in the interval $(0,5)$ where f fails to be differentiable, and let k denote the number of points in the interval $(0,5)$ where $f'(x) = 0$. (Here, as in the previous problem, the function $f(x)$ is shown in figure 2.) Then $2021^{n-k} =$

a) 2021 b) $\dfrac{1}{(2021)^2}$ c) $\dfrac{1}{2021}$ d) 1 e) None of the other answers provided.

11) Given the graph of $y = f(x)$ in Figure 2 and setting $L = \dfrac{1}{\lim\limits_{x \to 4^+} f'(x)}$, it follows that $2021^L + L^{2021} =$

a) Nothing since the proposed limit L does not exist. b) 2000 c) 2021 d) 2020
e) None of the other answers provided.

12) Let f be defined by the formula $f(x) = 2x + \sin x + \pi e^{\pi}$ and let $g(x) = f^{-1}(x)$. Use the fact that $g(\pi) = 0$ to determine the value of $g'(\pi)$.

a) $g'(\pi) = 1 + 3\pi$ b) $g'(\pi) = \dfrac{1}{3+\pi}$ c) $g'(\pi) = 2 + e\pi + \cos(1)$
d) $g'(\pi) = \dfrac{1}{2 + e\pi + \cos(1)}$ e) None of the other answers provided.

13) Given that a and b are two positive numbers satisfying the two equations

$$\lim_{n \to \infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^{2n} = 21$$
$$a^2 + b^2 = 58$$

We can conclude that the value of $|a - b|$ is

a) 6 b) 0 c) 42 d) 4 e) None of the other answers provided.

14) Suppose we are told that $\lim_{x\to\infty} f(x)\cdot\cos(x)$ exists. Which, if any, of the following statements can be true?

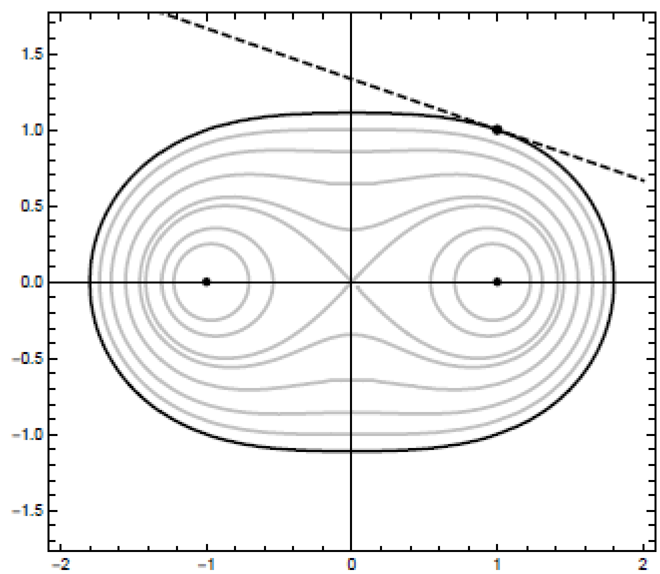
I. $f(x) = x^{-1}$
II. $f(x) = \tan x$
III. $f(x) = \cosh x - \sinh x$

a) II and III only b) I and III only c) I, II, and III d) I and II only e) None of the other answers provided.

15) If the function $f(x)$ is continuous for all real numbers is given by the formula $f(x) = \dfrac{x^2 - 7x + 12}{x - 4}$ for $x \neq 4$, then $f(4) =$

a) 1 b) -1 c) 0 d) $\dfrac{8}{7}$ e) Undefined

16) Several curves are shown below (some that look oval-shaped, others that look like indented eggs, and still others that cross themselves or are even disconnected).



Each of these curves is an example of a Cassini Oval*, and the one shown in black is given by the equation

$$\left((x+1)^2 + y^2\right)\cdot\left((x-1)^2 + y^2\right) = 5.$$

The tangent line depicted in the image above passes through the point $(1, 1)$. The slope of this line equals

[*Many Calculus students are familiar with standard ellipses given by equations such as $a^{-2}x^2 + b^{-2}y^2 = 1$, but are perhaps less familiar with the fact that these ellipses are the set of all points whose *sum* of distances to two fixed points (called the "foci") is constant. A Cassini Oval, on the other hand, is defined as the set of all points whose *product* of such distances is constant. In the figure above, the two foci are located at $(\pm 1, 0)$.]

a) -1 b) $-\dfrac{1}{2}$ c) -5 d) $-\dfrac{1}{3}$ e) None of the other answers provided.

17) Suppose $g(x)$ is a continuous function whose domain and range both equal the interval $[0,1]$. One can conclude that the graph of $y=g(x)$ must intersect the graph of $y=x^{2021}$ at one or more points by applying the Intermediate Value Theorem to which of the following functions?

a) $f(x)=g(x)x^{2021}$ b) $f(x)=\dfrac{g(x)}{x^{2021}}$ c) $f(x)=g(x)+x^{2021}$ d) $f(x)=g(x)-x^{2021}$
e) None of the other answers provided.

18) We are given that the following limit holds for *some* function $f(x)$: $\lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(1+\frac{4i}{n}\right)\frac{4}{n}=0.$

Of the following options provided below, which, if any, could equal the function $f(x)$?

a) $f(x)=2021(x-3)^2$ b) $f(x)=x^3-27$ c) $f(x)=\mathrm{e}^x$ d) $f(x)=x-3$
e) None of the other answers provided.

19) Given that $2000+\displaystyle\int_0^b \frac{63}{\pi(1+x^2)}\mathrm{d}x=2021$, the value of b is

a) $b=\sqrt{2}$ b) $b=1$ c) $b=\sqrt{3}$ d) $b=\dfrac{1}{\sqrt{3}}$ e) None of the other answers provided.

20) If the piecewise function $f(x)$ is defined for all real numbers x by $f(x)=\begin{cases}2x^2+4 & \text{if } x\le 1\\ 7-x & \text{if } x>1\end{cases}$

then which, if any, of the following statements are true?

a) $f(x)$ is continuous everywhere, $f(x)$ is differentiable everywhere, and $f(1)$ is a local maximum.
b) $f(x)$ is discontinuous and $f(1)$ is a local minimum.
c) $f(x)$ is continuous everywhere, $f(x)$ is not differentiable everywhere, and $f(1)$ is a local maximum.
d) $f(x)$ is continuous everywhere, $f(x)$ is not differentiable everywhere, and $f(1)$ is a local minimum.
e) None of the other answers provided.

21) The graph of the differentiable function $y=f(x)$ has a tangent line when $x=2$, and this line passes through the points $(1,1)$ and $(3,9)$. Based on this information determine which of the following statements is true.

a) $f(2)=3$ and $f'(2)=1$ b) $f(2)=9$ and $f'(2)=4$ c) $f(2)=5$ and $f'(2)=4$
d) $f(2)=1$ and $f'(2)=9$ e) None of the other answers provided.

22) The function $f(x)$ has a continuous second derivative, and its graph has a tangent line at the point $(1,3)$ given by $y=3$. If we are also told that $f''(1)=b$, then amongst the options provided below, which value of b allows us to conclude that $f(1)$ is a local minimum?

a) $b=\pi-\mathrm{e}$ b) $\ln\left(\dfrac{1}{2021}\right)$ c) $b=\mathrm{e}-\pi$ d) $b=0$
e) There are no values of b for which $f(1)$ is a local minimum since $x=1$ is not a critical number for f.

23) The definite integrals $I(k) = \int_1^k \frac{1}{x} - \sin(2\pi kx)\mathrm{d}x$ and $J(k) = \int_1^3 k^x + \cos(2\pi kx)\mathrm{d}x$ each depend on the value of the positive, whole number k. For which values of such k is the product $I(k)J(k)$ a whole number that is divisible by 3?

a) This is true for $k = 2021$ only.
b) This is not true for any values of k.
c) This is true for all odd values of k.
d) This is true for all values of k.
e) None of the other answers provided.

24) Figure 3 shows the graph of a function $y = f(x)$ that passes through the points $(a, 1)$ and $(75, 27)$. If we are told that

$$\int_a^{75} f(x)\mathrm{d}x + \int_1^{27} f^{-1}(y)\mathrm{d}y = 2021$$

then what is the value of the number a?

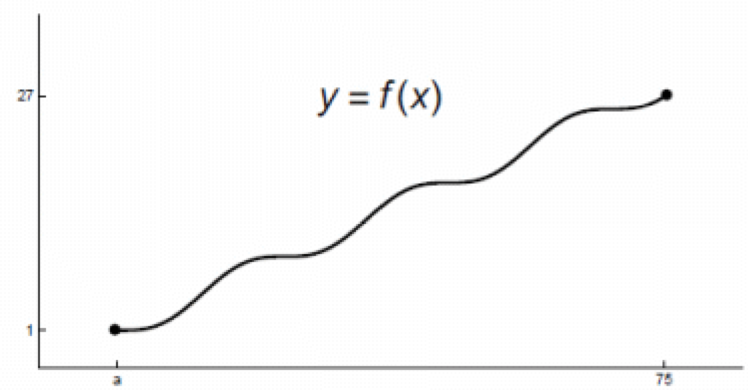


Figure 3

a) $a = 1$ b) $a = 2020$ c) $a = 15$ d) $a = 4$ e) None of the other answers provided.

25) Find an exact solution to the Ordinary Differential Equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x}{y}\right)^2$ with initial condition $y(0) = 1$.

a) $y = (\sqrt[3]{x} + 1)^3$ b) $y = x^3 + 1$ c) $y = \sqrt[3]{x^3 - 1}$ d) $y = \sqrt[3]{x^3 + 1}$
e) None of the other answers provided.

26) Consider an ellipse given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a \geq b > 0$. What is the maximum *product* of distances from the focii to a point on the curve?

a) $a^2 - b^2$ b) a^2 c) b^2 d) $2a$ e) None of the other answers provided.

27) The graph of f', the derivative of f, is the line shown in Figure 4 below.

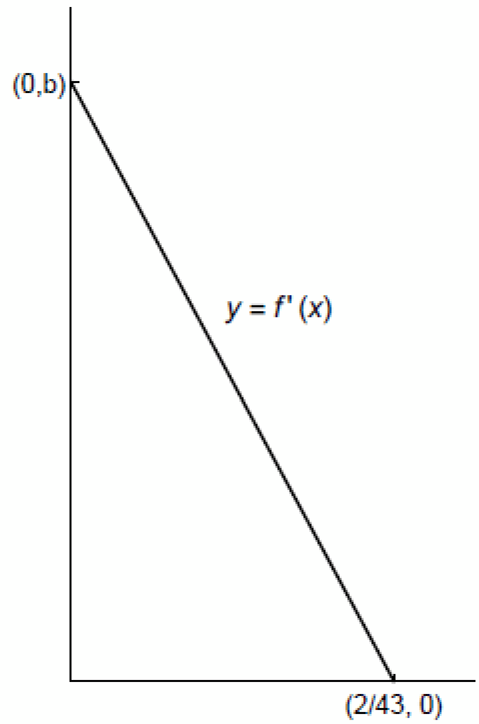


Figure 4: Plot of $f'(x)$

If $f(0) = 10$ and $f\left(\frac{2}{43}\right) = 57$, then what is the value of b?

a) $b = 10$ b) $b = 43$ c) $b = 2021$ d) $b = 2022$ e) None of the other answers provided.

28) The radius of a sphere is increasing at a rate proportional to the value of the radius. If the radius initially measures 3 cm and the radius equals 6 cm two seconds later, how large will the radius be after 8 seconds?

a) 12ln2 cm b) 12 cm c) $\ln(\sqrt{2})$ cm d) 48 cm e) None of the other answers provided.

29) Suppose $f(x)$ is everywhere continuous with known limits

$$\lim_{x \to 2} f(x) = 5$$

$$\lim_{x \to 2} f\left(f(x)\right) = 7$$

Which, if any, of the following conclusions hold?

a) $f(2) = 7$ b) $f(7) = 5$ c) $f(7) = 2$ d) $f(5) = 7$ e) No conclusions may be drawn.

30) Which functions $f(x)$ satisfy the following property: the average value of f over the interval $[0,b]$ equals $\dfrac{f(b)}{3}$ for every possible $b \geq 0$?

I. Any linear function whose graph passes through the origin.
II. Any quadratic function whose graph is a parabola with its vertex at the origin.
III. Any function that satisfies the separable ODE $2y = x\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

a) I, II and III b) I and II only c) II and III only d) I only
e) None of the other answers provided.

31) Bizarre as it may seem, the function $f(x) = \arctan(\sinh(x)) - \arcsin(\tanh(x))$ satisfies a lovely Ordinary Differential Equation (with initial condition $f(0) = 0$). Which equation does $f(x)$ satisfy?

a) $f'(x) = -f(x)$ b) $f'(x) = 0$ c) $f'(x) = \mathrm{e}^{-x^2}$ d) $f'(x) = 1$
e) None of the other answers provided.

32) Figure 5 shows the graph of f', the derivative of a function f. Based on this graph, determine which one, if any, of the following statements are true.

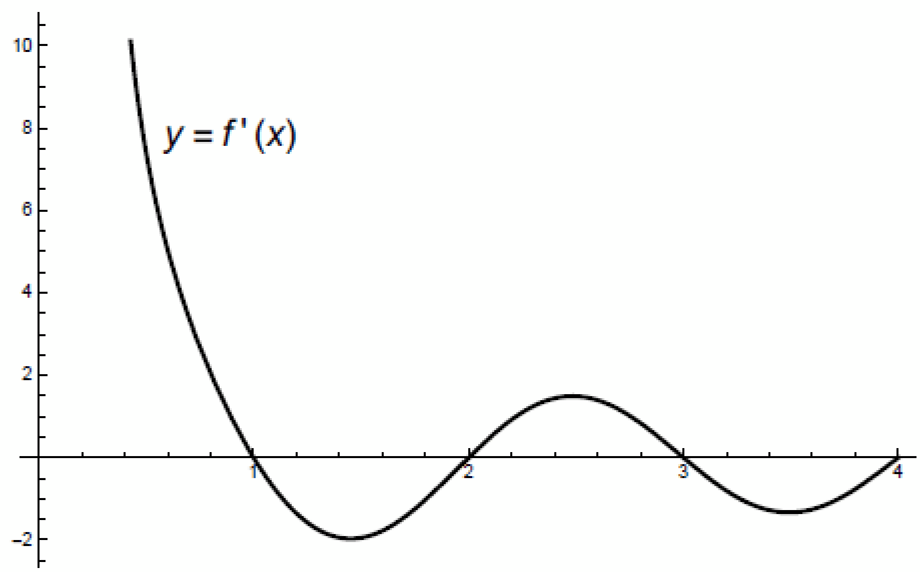


Figure 5: $y = f'(x)$

a) f is concave up on $(1,2) \cup (3,4)$ and concave down on $(0,1) \cup (2,3)$.
b) f is concave down on $(1,2) \cup (3,4)$ and concave up on $(0,1) \cup (2,3)$.
c) $f(1)$ and $f(3)$ are local minima for f.
d) f is increasing on the interval $(0,4)$.
e) None of the other answers provided.

33) Several integral expressions are written below. Which ones correspond to the area of the region shown in Figure 6?

I. $\int_{-1}^{1} x^2 - 2\cos\left(\frac{\pi x}{2}\right) - 1\, dx$

II. $\int_{-1}^{1} 2\cos\left(\frac{\pi x}{2}\right) - x^2 + 1\, dx$

III. $2\int_{1}^{2} \sqrt{y-1}\, dy + \frac{4}{\pi}\int_{2}^{4} \arccos\left(\frac{y}{2} - 1\right)\, dy$

IV. $2\int_{2}^{4} \sqrt{y-1}\, dy + \frac{4}{\pi}\int_{1}^{2} \arccos\left(\frac{y}{2} - 1\right)\, dy$

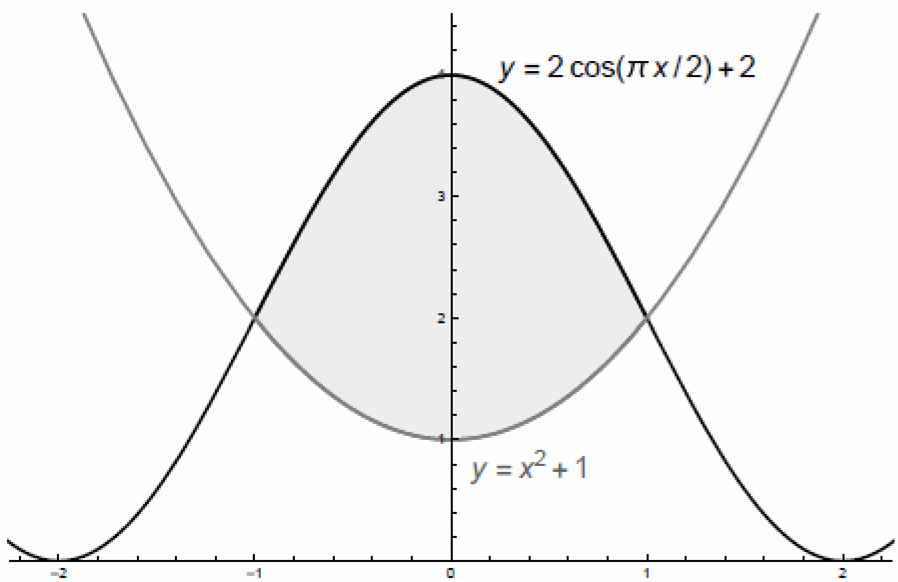


Figure 6

a) I, III and IV only b) I and IV only c) II and III only d) II and IV only e) I and III only

34) The top of a 25 foot ladder is sliding down a wall at a constant rate of 2 feet per minute, but this information is irrelevant to this question. What *is* relevant is the fact that atop this falling ladder stood a mathematician who painted on her wall an antiderivative for the function $(x^2 + 3x + 2)^{-1}$. Which beautiful function did she paint?

a) $\arctan(x+1)$ b) $-\frac{1}{2}(x^2 + 3x + 2)^{-2}$ c) $\ln(x^2 + 3x + 2)^{-2}$ d) $2x + 3$ e) $\ln\left|\frac{1+x}{2+x}\right|$

35) Let k denote the number of questions from this test that you answered correctly. The function $f(x) = \frac{xe^x}{k^2+1} + xe^k$ has a point of inflection when x equals which number?

a) $x = -k$ b) $x = -e$ c) $x = 0$ d) $x = -2$

e) The function has no points of inflection.