Calculus Exam - University of Houston Math Contest
January 29, 2022

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1) This problem uses two numbers: one of them is labelled $a > 0$, and it denotes the number of students currently reading this problem; the other, $b > 0$, is a to-be-determined positive real number. If we are told that

$$\lim_{x \to a} \frac{b^x - b^a}{x - a} = \left(\ln(2) + \frac{\log_b(3)}{\log_b(e)} + \frac{\log_a(337)}{\log_a(e)} \right) b^a$$

then which, if any, of the following is a true statement?

a) There are 2022 students currently reading this problem. b) $b = 2022$ c) $b = e^{2022}$
d) $b = \ln(2022)$ e) None of the other answers provided.

2) Suppose $f(x)$ and $g(x)$ are differentiable functions whose graphs both pass through the point $(2, 5)$; moreover, suppose the line tangent to the graph of $y = f(x)$ has slope -1 at this point and that the line perpendicular to the graph of $y = g(x)$ has slope $1/3$ at this point. Determine the value of $h'(2)$ where $h(x)$ is the function

$$h(x) = \frac{f(x)}{g(x)} + f(x)g(x).$$

a) $h'(2) = -2$ b) $h'(2) = -102/5$ c) $h'(2) = -98/5$
d) $h'(2) = -54/15$ e) None of the other answers provided.

3) The graph of a function's derivative, $f'(x)$, is shown in Figure 1.

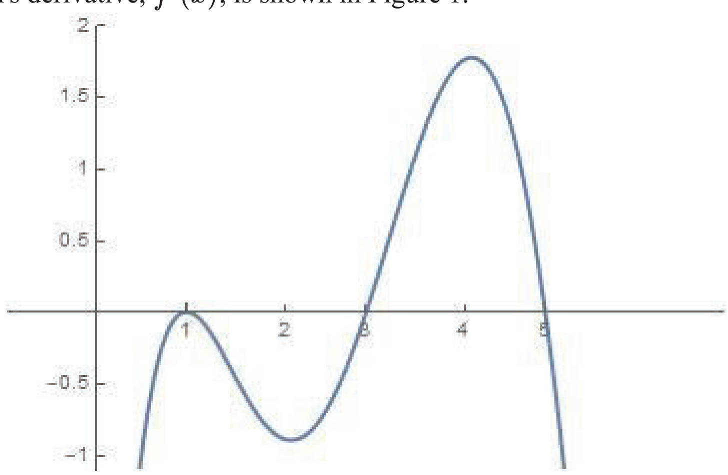


Figure 1: Plot of $y = f'(x)$

Based on the information from Figure 1, on which intervals, if any, is $f(x)$ both concave up and increasing?

a) $(0, 1)$ b) $(3, 5)$ c) $(2, 4)$
d) $(3, 4)$ e) There are no intervals along which $f(x)$ is concave up and increasing.

4) A 10 foot ladder is sliding down the wall of a rectangular room whose dimensions are 10 ft $\times$ 10 ft. The height of the ladder happens to be falling at a constant rate of 3 ft/sec. At different moments in time the area of the room lying above the sliding ladder takes on different values (see the images below).

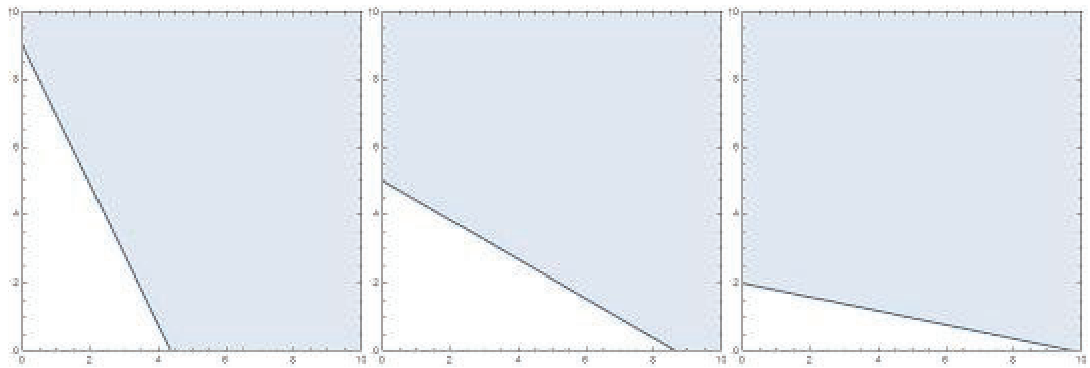


Figure 2: Initially upright Ladder sliding Down
(with area above shaded)

How quickly is the bottom of the ladder sliding to the right at the moment in time when the area above the ladder is minimal?

a) $\dfrac{3\sqrt{95}}{5}$ ft/sec.
b) The rate of change is undefined at this moment in time.
c) 3 ft/sec.
d) $3\sqrt{2}$ ft/sec.
e) None of the other answers provided.

5) For which value of a does the following inequality hold?

$$\lim_{x\to 0}\left(1+x\right)^{\frac{a}{x}} < a$$

a) There is no value of a that makes this inequality true.
b) $a = 1$
c) $a = 2022$
d) $a = 0$
e) None of the other answers provided.

6) At each point (x, y) along a given differentiable curve there is a tangent line with slope $\sec(y)\sec^2(x)$. If the curve passes through the point $(0, 0)$, what is the x-coordinate of the curve when $y = \pi/2$?

a) $x = 1$
b) $x = \pi/6$
c) $x = \pi/4$
d) $x = \pi/3$
e) $x = \pi/2$

7) This problem makes use of three numbers, a, b and c, each determined by the equations below.

$$\frac{d^2}{dx^2}\Big(\sin(ac\,x)\Big) = -196\sin(ac\,x)$$

$$\int_1^b \frac{dx}{x^2} = \frac{2}{3}$$

$$\lim_{x\to c} \frac{\sin(bx-6)}{bx-6} = 1$$

Compute the value of the following expression:

$$\frac{c\cdot 10^b + 3a^{c-1} + 1}{\big(b\cdot 10^c + b\cdot 10^{b-c} + a\big)} =$$

a) 6 b) 1 c) 14 d) $\dfrac{1011}{1505}$ e) None of the other answers provided.

8) Consider the parabola $y = a^2 - x^2$ where a is a real number. An image of such a parabola is shown in the figure below, along with two regions enclosing A_1 and A_2 units of area, respectively.

For which real numbers a, if any, is A_1 twice as large as A_2?

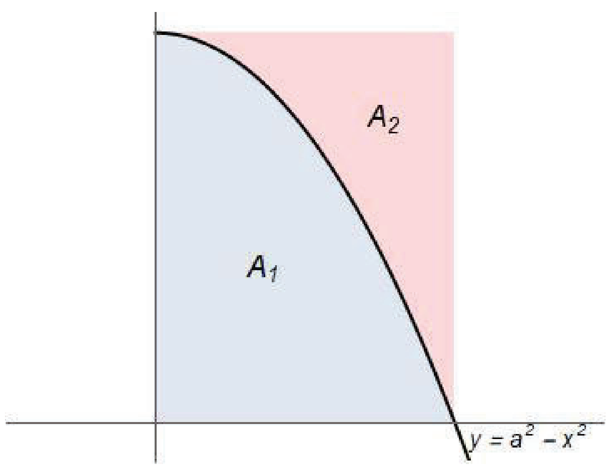


Figure 3

a) This is only true for $a = \pm\sqrt{3}$.
b) This is only true for $a = 2022$.
c) This is only true for $a = 0$.
d) This is true for all real numbers $a \in (-\infty, \infty)$.
e) This is only true for $a = \pm 1$.

9) The value of the limit

$$\lim_{h \to 0} \frac{\sinh(2x) - \sinh(2x+2h)}{h}$$

depends on the real number x. The largest value of this limit occurs when

a) $x = 0$ b) $x = 1$ c) $x = \ln(\sqrt{2})$ d) $x = -4$ e) None of the other answers provided.

10)

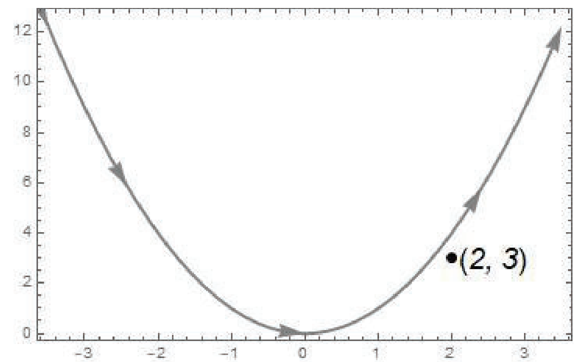


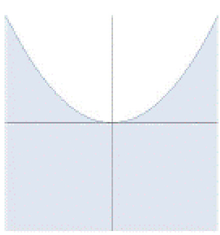
Figure 4: The Rocket Path Traces Out the Graph $y = x^2$

The figure above shows the path along which astronaut Emmy Ripley is flying her rocket ship (she is steering carefully so that the rocket always moves from left to right). At some point during her journey she will shut off the rocket's engines and let the ship drift along whatever fixed direction in which it happens to be pointing.

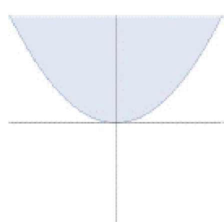
A distress call has emerged from planet LV-426 which is located at point $(2, 3)$. At which point (x, y) along Dr. Ripley's parabolic path should she begin to drift so that she reaches the planet?

a) $(1/2, 1/4)$ b) $(3, 9)$ c) $(\sqrt{3}, 3)$
d) $(1, 1)$ e) None of the other answers provided.

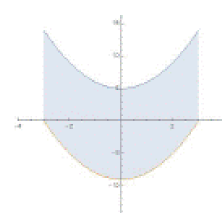
11) This question also uses Figure 4 and information about Dr. Ripley's travels. Several images are shown below.



(a) Region 1



(b) Region 2



(c) Region 3

Which depicted region, if any, shows all of the locations to which Dr. Ripley can drift?

a) Dr. Ripley can drift to *any* point in the plane. b) Region 3
c) Region 2 d) Region 1 e) None of the other answers provided.

12) Dr. Harvey Dent plays a peculiar Calculus game using his fair coin. Whenever he has a function $f(x)$ in front of him he'll flip his coin, and if it lands heads up he differentiates the function to produce $f'(x)$. Of course, whenever his coin lands tails up he anti-differentiates it to produce the function $F(x)$ and always makes certain that $F(0) = 1$.

What is the probability that after four coin flips Dr. Dent produces the same function with which he started, $f(x) = x + x^2 + e^x$?

a) 1/4 b) 3/16 c) 1/2 d) 3/8 e) None of the other answers provided.

13) If the function $f(x)$ is continuous for all real numbers and is given by the formula $f(x) = \dfrac{2x^3 - 18x^2 + 36x}{x-3}$ when $x \neq 3$, then $f(3) =$

a) $\dfrac{2022}{6} - 336$ b) $-\dfrac{2022}{337} \cdot 3$ c) 2022 d) 2022 − 2022 e) Undefined

14) For this question the constant a denotes your current age (so that if you are sixteen years old we would set $a = 16$). Let $(c, 0)$ denote the the point where the line tangent to $y = \ln(x)$ at $x = a$ intercepts the x-axis, and let A denote the area under the graph $y = \ln(x)$ along the interval $[1, a]$ as depicted in Figure 6.

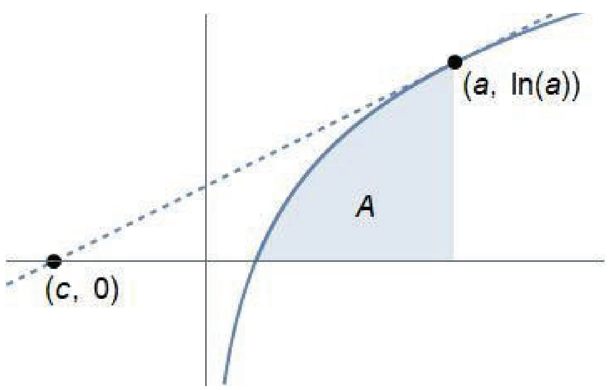


Figure 6

The value $\left|\dfrac{A}{c-1}\right|$ equals

a) 1 b) e c) a d) $\ln(a)$ e) None of the other answers provided.

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15) In Calculus courses the world over students are commonly taught to relate a function's second derivative, $f''(x)$, with how "curvy" or "bendy" its graph is. However, these courses have been (subtly) misinforming you the whole time!

The notion of *concavity* is, in fact, distinct from a rigorous notion of *curvature*, and you can see this by considering the function x^2. Its second derivative is constant, but its graph certainly does not appear to be "constantly curved" (the way a circle appears)!

As it turns out, the correct definition of the **curvature of a graph** $y = f(x)$ involves a more complicated formula we designate by $\kappa_f(x)$ and define as

$$\kappa_f(x) = \frac{f''(x)}{\left(1 + (f'(x))^2\right)^{3/2}}.$$

This formula accurately captures how curved the graph of $y = f(x)$ is at a particular point (x, y).

When we add up all of these curvature values along an interval in f's domain, the resulting quantity is called **the total curvature of** $f(x)$.

For this problem, compute the total curvature of $f(x) = \cosh x$ along the interval $[0, ln(3)]$.

$$\int_0^{\ln(3)} \kappa_{\cosh}(x)\, dx =$$

a) $4/5$
b) $2/3$
c) 0
d) $4/3$
e) None of the other answers provided.

16) We are given that the following limit holds for *some* to-be-determined, positive number $b > 0$:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n}{\pi n^2 + \pi i^2 b^2} = \frac{1}{4b}.$$

Of the following options provided below, which, if any, could equal the constant b?

a) There is no value of b that makes this equation true.
b) $b = \pi/4$
c) $b = 1$
d) $b = e/2$
e) None of the other answers provided.

17) This problem uses the function $y = x^2$ to construct different triangles. Every real number a gives rise to a triangle with vertices $(a, a^2), (0, 1/4)$, and $(a, -1/4)$, as shown in Figure 7.

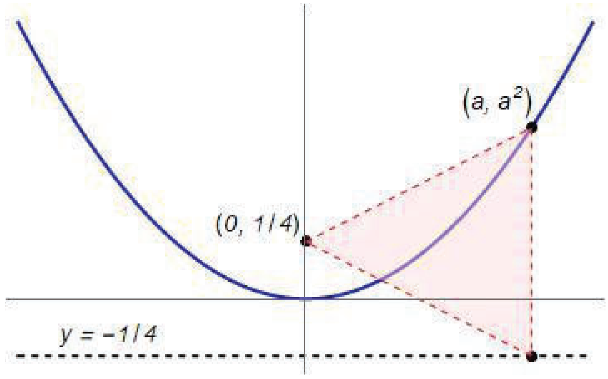

Figure 7: Making Triangles using $y = x^2$

If a is chosen so that the resulting triangle is equilateral, then $|y'(a)| =$

a) 1
b) $\sqrt{2}$
c) $\sqrt{3}$
d) $\dfrac{1}{\sqrt{3}}$
e) $\dfrac{1}{\sqrt{2}}$

18) Consider the following claim: "There exists at least one number $c \in (0, 1)$ satisfying

$$\sum_{n=1}^{2022} n\,(-1)^{n-1}\,c^{n-1} = 0."$$

Which, if any, of the explanations provided below can be used to justify this claim?

a) One can apply the Mean Value Theorem to the polynomial $f(x) = \sum_{n=1}^{2022} (-1)^{n-1}\,x^n$ on the interval $[0, 1]$.

b) One can apply the Intermediate Value Theorem to the polynomial $f(x) = \sum_{n=1}^{2022} (-1)^{n-1}\,x^n$ on the interval $[0, 1]$.

c) The stated claim is false and so cannot be justified.

d) One can apply the Intermediate Value Theorem to the polynomial $f(x) = \sum_{n=1}^{2022} n\,(-1)^{n-1}\,x^{n-1}$ on the interval $[0, 1]$.

e) None of the other answers provided.

19) Everyone taking this test is very familiar with the fact that whole numbers come in two flavors: *even* and *odd*. But did you know that its possible to extend this definition so as to talk about *how* even or odd a number is? In fact, we can even talk about *how* even or odd any real number is!

Here is one way to do this: we set up a function $E(x)$ that measures how even or odd the real number x using values between 0 and 1. This is accomplished by using the formula

$$E(x) = \text{ the distance from } x \text{ to the nearest odd whole number.}$$

For example, the even number -4 has an "even rating of 1" since $E(-4) = 1$ (where we can use either -3 or -5 to compute the distance from -4 to a nearest odd), while $E(\pi) = .1415926\ldots$ so π is "not very even," and $E(7) = 0$ meaning the number 7 is "not even at all."

Let A_n denote the average value of $E(x)$ over the interval $[0, n]$ (where n is a positive whole number). Then $\lim\limits_{n\to\infty} A_n =$

a) 1 b) This limit does not exist. c) 0 d) 1/2 e) None of the other answers provided.

20) This problem uses the function $y = 2/x$ to construct different triangles. Every $x \in (0, \infty)$ gives rise to a triangle formed by the coordinate axes and the line tangent to the graph at $(x, 2/x)$, as shown in Figure 8.

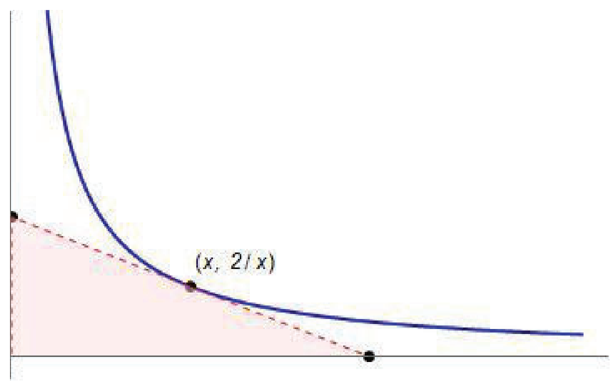

Figure 8: Triangle formed using $y(x) = 2/x$

Let $A(x)$ denote the area of each such triangle. Which differential equation does $A(x)$ satisfy?

a) $A'(x) = 4A(x)$
b) $A'(x) = 4x$
c) $A'(x) = -4x$
d) $A'(x) = 0$
e) None of the other answers provided.

21) If the piecewise function $f(x)$ is defined for all real numbers x by

$$f(x) = \begin{cases} 3x^2 + 1 \text{ if } x \leq 3 \\ 34 - 2x \text{ if } x > 3 \end{cases}$$

then which, if any, of the following statements are true?

a) $f(x)$ is discontinuous and $f(3)$ is a local minimum.
b) $f(x)$ is continuous everywhere, $f(x)$ is not differentiable everywhere, and $f(3)$ is a local maximum.
c) $f(x)$ is continuous everywhere, $f(x)$ is differentiable everywhere, and $f(3)$ is a local maximum.
d) $f(x)$ is continuous everywhere, $f(x)$ is not differentiable everywhere, and $f(3)$ is a local minimum.
e) None of the other answers provided.

22) The graph of the second derivative of a function, $y = f''(x)$, is shown below in Figure 9.

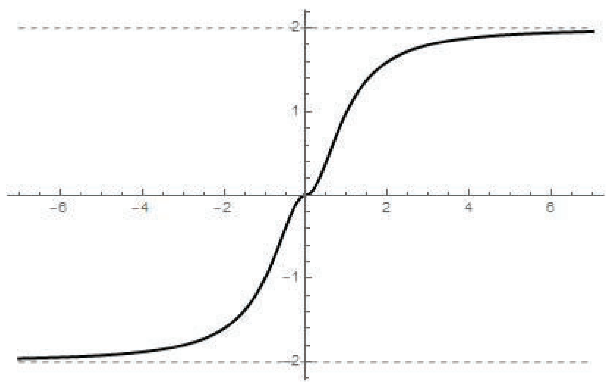

Figure 9: $y = f''(x)$

Based on this image compute

$$\left(\lim_{x\to\infty} \frac{f'(x)}{x}\right) \cdot \left(\lim_{x\to-\infty} -\frac{f'(-x)}{x}\right) =$$

a) 4 b) -2 c) 2 d) -4 e) 0

23) The function $f(x)$ has a continuous second derivative, and its graph has a tangent line at the point $(20, 22)$ given by $y = 22$. If we are also told that $f''(20) = b$, then amongst the options provided below, which value of b allows us to conclude that $f(20)$ is a local maximum?

a) $b = 2022$
b) None of the other answers provided.
c) There are no values of b for which $f(20)$ is a local maximum since $x = 20$ is not a critical number for f.
d) $b = \ln(2022)$
e) $b = 2022\tan(\pi/4)$

24) The graph of $y = f(x)$ is shown below.

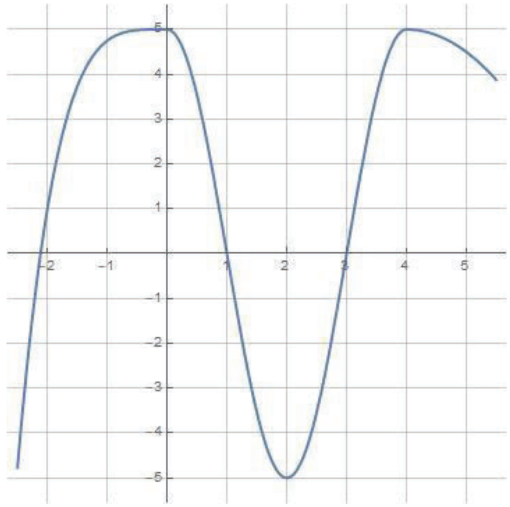

Figure 10: $y = f(x)$

Based on this graph one finds that

$$\int_{-2}^{0} x \left| f'\left(x^2\right) \right| dx =$$

a) -10 b) $22/3$ c) $-22/3$ d) 10 e) None of the other answers provided.

25) Suppose that $f(x)$ is a positive, increasing, differentiable function defined for all real numbers and suppose the function $g(x) = f(x)^x$ satisfies $g'(0) = 0$. Which, if any, of the following statements necessarily follows?

a) $f(0) = 1$ b) $f'(0) = 1$ c) $f'(0) = 1/e$ d) $f(0) = e$
e) None of the other answers provided.

26) Evaluate $y(\sqrt{15})$ where $y(x)$ is the unique solution to the Ordinary Differential Equation

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$$

with initial condition $y(0) = 0$.

a) $y\left(\sqrt{15}\right) = \dfrac{\sqrt{15}}{4}$ b) $y\left(\sqrt{15}\right) = 4$ c) $y\left(\sqrt{15}\right) = 4\sqrt{15}$
d) $y\left(\sqrt{15}\right) = \sqrt{15}$ e) None of the other answers provided.

27) Suppose we are told that

$$\frac{d}{dx}\left(\int_{2022}^{\cos x} f(t)\,dt\right) = -\cos(\cos x)\sin x.$$

Then $f(0) =$

a) 0 b) 1 c) π d) -1 e) None of the other answers provided.

28) The graph of the function $f(x)$ is shown in Figure 11, and it consists of four circular arcs (each subtending a central angle of 90°). The radii of these arcs follow a particularly nice pattern, namely the radius of the n-th arc equals $1/2$ the radius of the previous $n-1$-th arc.

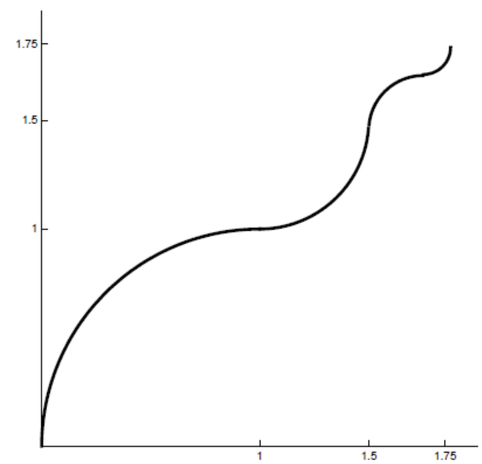

Figure 11: $y = f(x)$

Evaluate the definite integral

$$\int_0^{3/2} f^{-1}(y)\,dy =$$

a) $1 - \frac{\pi}{4}$ b) $\frac{3}{4} + \frac{\pi}{16}$ c) $\frac{3}{2} - \frac{3\pi}{16}$ d) $\frac{9}{4}$ e) None of the other answers provided.

29) A rectangular, square-base box containing 16 cubic inches of volume has a shrinking height, one that changes at a constant rate of 1 inch per second. Assuming that the length and width of the square base change in such a way so that the box's volume remains fixed, what will be the rate of change in the base length when the box's height is 4 inches?

a) $1/4$ inches per second b) $-1/4$ inches per second c) 2 inches per second
d) -2 inches per second e) None of the other answers provided.

30) Suppose the function $f(x)$ has a tangent line at $x = 9$ given by $y = 4(x-9)+2$. Then

$$\lim_{x \to 9} \frac{f(x) - 2}{\sqrt{x} - 3} =$$

a) $4\sqrt{3}$ b) -16 c) 24 d) 4< e) 8

31) This problem uses the function $y = \frac{\sqrt{8}}{3} \cdot \sqrt{9 - x^2}$ to construct different triangles. Every $x \in (-3, 3)$ gives rise to a triangle with vertices $(-1, 0), (x, y(x))$ and $(1, 0)$, and we let $P(x)$ denote its perimeter.

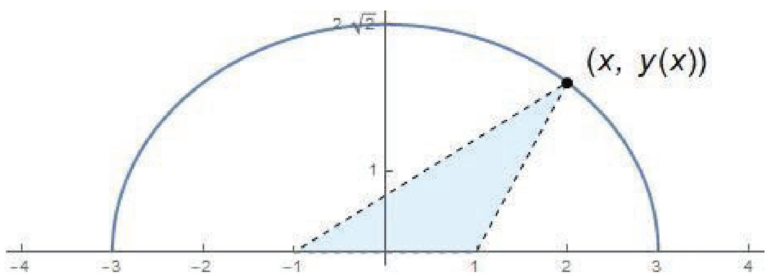

Figure 12: A triangle with perimeter $P(2)$

Evaluate the following expression:

$$\int_0^3 P(x)\, dx =$$

a) 24 b) 64 c) $3\pi\sqrt{2}$ d) 9 e) None of the other answers provided.

32) **TIEBREAKER QUESTION:** Recall the well known "floor", $\lfloor\ \rfloor$, and "ceiling" $\lceil\ \rceil$ functions:

$$\lfloor x \rfloor = \text{the largest integer } \leq x$$
$$\lceil x \rceil = \text{the smallest integer } \geq x$$

Evaluate the following definite integral:

$$\int_0^\pi \lfloor \cos x \rfloor^{2022} + \lceil \sin x \rceil^{2022}\, dx =$$

a) 0 b) $\pi/2$ c) $3\pi/2$ d) π e) None of the other answers provided.