Calculus Exam - University of Houston 2023 Math Contest
January 28, 2023

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1) Suppose we are told that the following limit holds: $\lim_{x \to a} \dfrac{\sin(ax - a^2)}{x - a} = \sin(a)$, where a is some real number. Which, if any, of the following is a true statement?

a) $a = 0$ b) This equation is true for all possible values of $a \in \mathbb{R}$ c) $a = 2023$
d) $a = \pi$ e) There is no value of a that makes this equation true

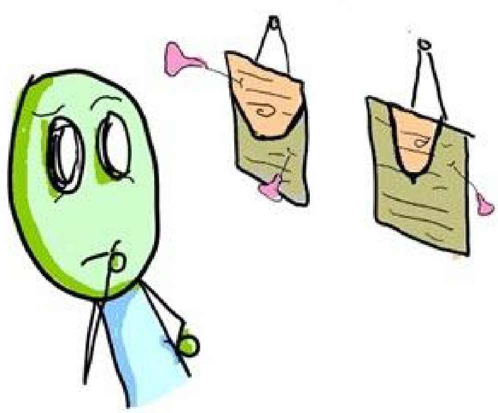
2) Suppose the function $f(x)$ has a graph that passes through the point $(0,0)$, and that, for every real number $x \neq 0$, $\dfrac{1291\cos^4 x + 731\cos^2 x + 1}{x} \leq \dfrac{f(x)}{x^2} \leq \dfrac{17^{x+2} \cdot 7^{x+1} + 17^{2-x} \cdot 7^{1-x}}{2x}$. What, if anything, can be concluded about $f'(0)$?

a) We can conclude $f'(0) = 0$
b) We can conclude $f'(0) = 2023$
c) We can conclude $f'(0) = 1$
d) We can conclude $f'(0)$ does not exist
e) We cannot make any conclusions about $f'(0)$

3) Consider the following definite integral: $I(a,b) = \int_a^{a^b} \dfrac{dx}{x \ln x}$ where $a > 1$ and $b > 0$ are arbitrary real numbers. If we are told that $\lim_{a \to \infty} I(a,b) = 1/2$ then what, if anything, can we conclude?

a) We can conclude $a = e^2$, but we cannot conclude anything about the value of b
b) We can conclude $a = 2023$ and we can conclude $b = 1$
c) We cannot conclude anything about the value of a, but we can conclude $b = \sqrt{e}$
d) We can conclude $a = e$ and we can conclude $b = 2023$
e) We can conclude $a = b = e^2$

4) The curve $y = x^2$ is painted on a square dartboard so that its apex, $(0,0)$, is located at the board's center. A clever mathematician calculated the probability that a randomly-thrown dart will land in the region above the parabola, and she found this probability to be $867/2023$.



Determine the dartboard's dimensions (we'll ignore units).

a) The dartboard is a $(98/81) \times (98/81)$ square
b) The dartboard is a $(6/7) \times (6/7)$ square
c) The dartboard is a $(18/7) \times (18/7)$ square
d) The dartboard is a $(324/49) \times (324/49)$ square
e) None of the other answers provided

5) Suppose we have a differentiable function $f(x)$ and we know that when $x = 17$ the graph of $g(x) = f(x)^{f(x)}$ has as its tangent line $y = 51x - 840$. Determine the values of $f(17)$ and $f'(17)$.

a) $f(17) = 3$ and $f'(17) = \dfrac{51}{ln3+1}$

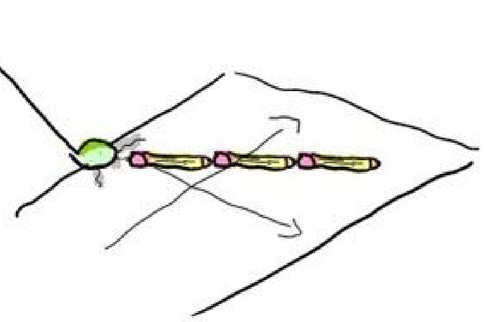
b) $f(17) = 1996$ and $f'(17) = 2023$

c) $f(17) = 3$ and $f'(17) = \dfrac{17}{9(ln3+1)}$

d) $f(17) = 27$ and $f'(17) = 51$

e) None of the other answers provided

6) A clever math student has completed their boring AP Calculus Exam, and, to pass the time, they explored different ways to arrange five different pencils. Specifically, they treated each pencil as a line segment and joined them to form graphs of continuous functions.



Suppose the student wishes to create a graph of a function $y = f(x)$ that has a horizontal tangent line at one (or more) points. What is the maximum number of points at which their function can be non-differentiable (excluding any endpoints of the function's domain)?

a) 1 b) 3 c) 2 d) 4 e) None of the other answers provided

7) Suppose we have an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ that enjoys the additional property that, for all real numbers x, $f(3x) = 27f(x)$. Which, if any, of the following statements is true?

a) $f(0) = f'(0) = f''(0) = 0$, but the value of $f'''(0)$ cannot be determined.

b) $f(0) = f'(0) = 0$, but the value of $f''(0)$ cannot be determined.

c) No information about $f(0)$ or any of its derivatives at 0 can be determined.

d) $f(0) = f'(0) = f''(0) = f'''(0) = 0$, but the value of $f^{(4)}(0)$ cannot be determined.

e) None of the other answers provided

8) Suppose we have a continuous function $f : \mathbb{R} \to \mathbb{R}$ that enjoys the additional properties: $f(3x) = 27f(x)$ and $\int_0^3 f(x)\,dx = 9$. Determine the value of the definite integral $\int_1^3 f(x)\,dx$

a) $\dfrac{1}{9}$ b) $\dfrac{80}{9}$ c) 8 d) 1 e) None of the other answers provided

9) Consider the following limit:

$$\lim_{x\to\infty} \frac{2023x}{289x + 7\cos x}$$

Which of the following statements is true?

a) The limit equals 7, and L'Hôpital's Rule *can* be used to compute it.

b) The limit equals 7, but L'Hôpital's Rule *cannot* be used to compute it

c) The limit equals 289, and L'Hôpital's Rule *can* be used to compute it.

d) The limit equals 289, but L'Hôpital's Rule *cannot* be used to compute it.

e) The limit does not exist.

10) The expression $\frac{1}{2023}\left(\sqrt{\frac{1}{2023}}+\sqrt{\frac{2}{2023}}+\sqrt{\frac{3}{2023}}+\cdots+\sqrt{\frac{2022}{2023}}+1\right)$ is a Riemann Sum approximation for

a) $\int_0^1 \sqrt{\frac{x}{2023}}\,dx$

b) $\frac{1}{2023}\int_0^{2023} \sqrt{x}\,dx$

c) $\frac{1}{2023}\int_0^1 \sqrt{\frac{x}{2023}}\,dx$

d) $\int_0^1 \sqrt{x}\,dx$

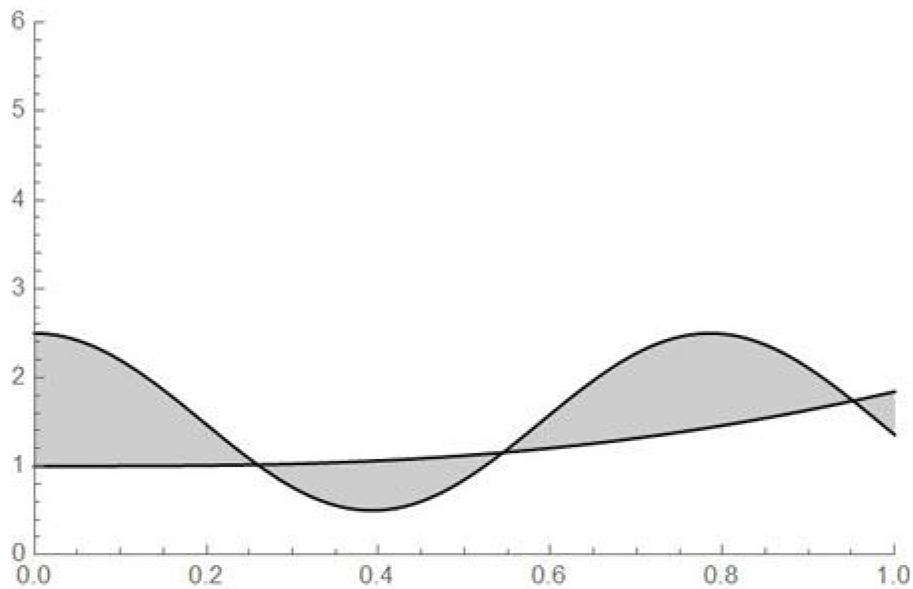
e) None of the other answers provided

11) For this question $f(x)$ and all of its infinitely many derivatives have $\mathbb{R}$ as their domains. Moreover, the graphs of both $f(x)$ and $f'(x)$ have tangent lines at $x=7$ that pass through the point $(20,23)$. If we also know that f has a point of inflection at $x=7$, then what is the value of $f(7)$?

a) $f(7)=-345$ b) $f(7)=-276$ c) $f(7)=-299$ d) $f(7)=2023$

e) None of the other answers provided

12) The distance between two points in the plane is calculated using the familiar 'distance formula,' but did you know that mathematicians also have ways of talking about 'the distance between two functions?' One such way is to use an interval of real numbers -- like $[0,1]$ -- and measure the areas of the regions trapped between the functions' graphs, like this



$$\text{dist}\left(f,g\right)=\int_0^1 |f(x)-g(x)|\,dx$$

Use this notion of distance to answer the following question: of all the lines through the origin, which one is closest to the function $g(x)=1-x$?

a) $f(x)=-x\sqrt{2}$ b) $f(x)=x$ c) $f(x)=\left(\sqrt{2}-1\right)x$ d) $f(x)=-x$ e) $f(x)=\frac{x}{1+\sqrt{3}}$

13) Suppose we have four functions, $f(x), g(x), h(x)$, and $j(x)$ that are each continuous on the closed interval $[0,4]$ and twice-differentiable on the open interval $(0,4)$. Exactly one of these functions has a negative first derivative and a positive second derivative. Based on the tables of values shown below, determine the function with these properties.

x	$f(x)$
1	10
2	9
3	8
4	7

x	$g(x)$
1	2023
2	−2023
3	2022
4	1980

x	$h(x)$
1	2023
2	23
3	0
4	−1

x	$j(x)$
1	7
2	119
3	289
4	2023

a) $h(x)$ must be the function with a negative first derivative and a positive second derivative.
b) $f(x)$ must be the function with a negative first derivative and a positive second derivative.
c) $j(x)$ must be the function with a negative first derivative and a positive second derivative.
d) $g(x)$ must be the function with a negative first derivative and a positive second derivative.
e) There is not enough information to determine which of the functions must have a negative first derivative and a positive second derivative.

14) Let $a \in \mathbb{R}$ be a fixed real number and consider the function $f(x)$ defined below:

$$f(x) = \begin{cases} 20x^2 + 23 & \text{if } x < 1 \\ 20x^2 + 2ax + 3 & \text{if } x \geq 1 \end{cases}$$

Determine the value of a, if any, so that the following conditions hold:

$$\text{(I)} \lim_{x \to 1^-} f(x) = 43 = \lim_{x \to 1^+} f(x)$$
$$\text{(II)} \lim_{h \to 0^-} \frac{43 - f(1+h)}{h} = 40$$
$$\text{(III)} \lim_{h \to 0^+} \frac{43 - f(1+h)}{h} = 50$$
$$\text{(IV)}\ f''(1) = 40$$

a) $a = 2023$ b) $a = 10$ c) There is no value of a that satisfies all four properties.
d) $a = 0$ e) $a = 1$

15)

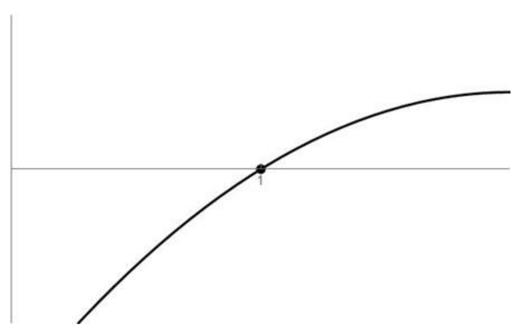


Figure 1: Graph of $y = f(x)$

The graph of a twice-differentiable function, $f : \mathbb{R} \to \mathbb{R}$, is shown in the provided image. Which of the following is true?
a) $f(1) < f'(1) < f''(1)$ b) $f''(1) < f(1) < f'(1)$ c) $f'(1) < f(1) < f''(1)$
d) $f''(1) < f'(1) < f(1)$ e) $f'(1) < f''(1) < f(1)$

16) The differentiable curve determined by the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{2023}$$

is shown below (where we have also assumed both $x < 0$ and $y < 0$).

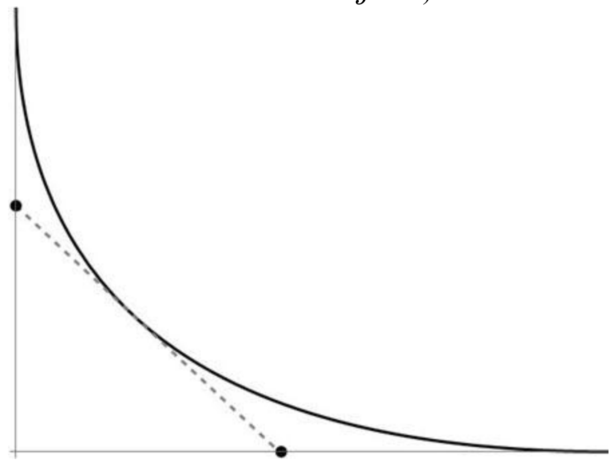


Figure 2: Curve for $\sqrt{x} + \sqrt{y} = \sqrt{2023}$ and a tangent line

The tangent line to an arbitrary point along the curve has x- and y-intercepts $(0, a)$ and $(b, 0)$. At which points is it true that $a + b = 2023$?

a) This is true only at the points $\left(\frac{2023}{4}, \frac{2023}{4}\right)$ and $(289, \sqrt{2023} - 17)$

b) There are no points along the curve for which this is true.

c) This is true only at the point $\left(\frac{2023}{4}, \frac{2023}{4}\right)$

d) This is true for *all* points along the curve.

e) None of the other answers provided.

17) A continuous function, $f : \mathbb{R} \to \mathbb{R}$, has an interesting property: for every real number t the limit of its average value along the interval $[t, t + h]$ equals $cos\ t$, and this limit is taken with h tending to zero. What, if anything, can we conclude about the function f?

a) It follows that f(x) is constant.

b) It follows that f(x) = sin x

c) It follows that f(x) = cos x

d) It follows that f(x) = -sin x

e) None of the other answers provided.

18) Suppose we are told that the following limit holds:

$$\lim_{h \to 0} \frac{(1 + h)^p - 1}{h} = \pi$$

What is the value of p?

a) p=2

b) p = π

c) p=2/π

d) p=2π

e) None of the other answers provided.

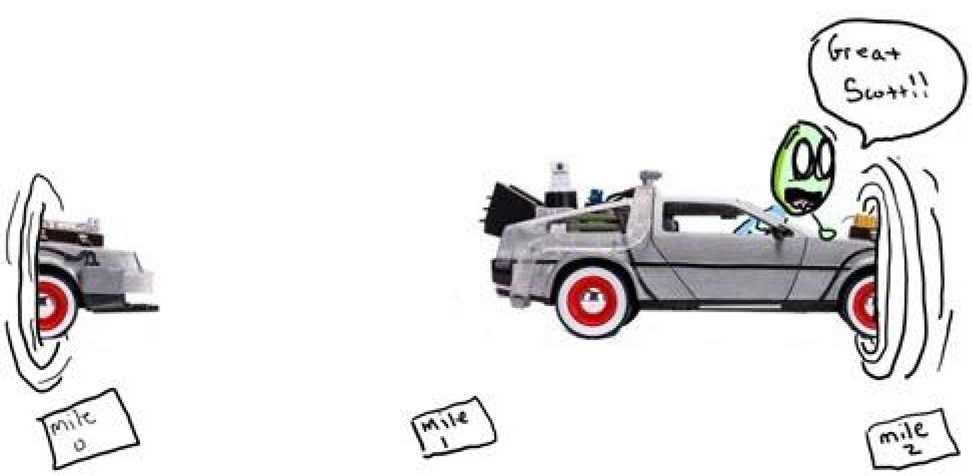
19) Consider the function

$$Q(x) = \int_0^x \left(q(t) + t^{22}\right)\, dt$$

If we are told that $Q(x)$ is a degree-23 polynomial, then which, if any, of the following statements is true?
a) q(t) is a polynomial, and its degree is less than or equal to 22.
b) q(t) may or may not be a polynomial (where constants are considered degree-0 polynomials), and if it is a polynomial, then its degree is less than or equal to 22.
c) q(t) is a polynomial, and its degree is greater than or equal to 23.
d) q(t) may or may not be a polynomial (where constants are considered degree-0 polynomials), and if it is a polynomial, then its degree is greater than or equal to 22.
e) None of the other answers provided.

20) If $x + 11y = 23$ is the equation for the *normal line* to the graph of $y = f(x)$ when $x = 1$ then $f(1) + f'(1)$ =
a) 13
b) -13
c) -9
d) $\dfrac{21}{11}$
e) None of the other answers provided.

21) A vehicle is being driven along a highway so that at any time $t \geq 0$ its velocity is given by $v(t) = t^2$ miles/minutes. The journey begins at mile marker 0, but there is a wormhole located at mile marker 2 that instantly relocates the vehicle *back* to mile marker 0!



The driver keeps going, passing mile markers 0, 1 and 2 over and over again. When is mile marker 1 passed for the 5th time?
a) This happens 2 minutes into the vehicle's journey.
b) This happens $(15)^{1/3} \approx 2.466$ minutes into the vehicle's journey.
c) This happens $\dfrac{5^3}{3} \approx 41.667$ minutes into the vehicle's journey.
d) This happens 3 minutes into the vehicle's journey.
e) None of the other answers provided.

22) ('Feynman Integration' aka 'Differentiating Under the Integral') The point of this question is to introduce a (likely) *new* technique of integration, one popularized by physicist Richard Feynman and not often taught in standard Calculus courses -- *especially* not ones based on an AP exam. Let's use this technique on the definite integral

$$\int_0^1 \frac{x-1}{\ln x}dx$$

The careful reader may be suspicious that our integrand is undefined at the points $x=0$ and $x=1$, but this is easily overcome since the integrand has one-sided limits here.
Feynman's approach to this integral proceeds as follows:

Step 1. Introduce a new variable t and the new function

$$I(t)=\int_0^1 \frac{x^t-1}{\ln x}dx$$

Observe that $I(1)$ is the definite integral we wish to calculate, and that $I(0)=0$.

Step 2. Compute a formula for $I'(t)$ by first 'differentiating under the integral sign'

$$I'(t)=\frac{d}{dt}\left(\int_0^1 \frac{x^t-1}{\ln x}dx\right)=\int_0^1 \frac{d}{dt}\left(\frac{x^t-1}{\ln x}\right)dx$$

The resulting integral should be easy to compute, leaving an explicit formula for $I'(t)$

Step 3. Use your formula for $I'(t)$ and the initial condition $I(0)=0$ to find an explicit formula for $I(t)$. Lastly, evaluate $I(1)$.

What is the value of our original definite integral, $I(1)$?
a) $I(1)=3\sqrt{2}$
b) $I(1)=\dfrac{-11}{6}$
c) $I(1)=1$
d) $I(1)=\ln(2)$
e) None of the other answers provided.

23) Use the function $F(x)=\int_2^x \sqrt{1+t^3}\,dt$ to compute the limit

$$\lim_{x\to 2}\frac{F(x)}{\sqrt{x}-\sqrt{2}}=$$

a) $6\sqrt{2}$
b) 0
c) 3
d) $2\sqrt{3}$
e) The limit does not exist.

24) Two circles centered at different points along the x-axis are expanding about their fixed centers. The first circle's perimeter is increasing at a constant rate of 6π cm/sec, having begun with an initial perimeter of 2π cm. The second circle's *area* is increasing at a constant rate of $3\pi\ \text{cm}^2/\text{sec}$, itself having begun with an initial area of $\pi\ \text{cm}^2$.

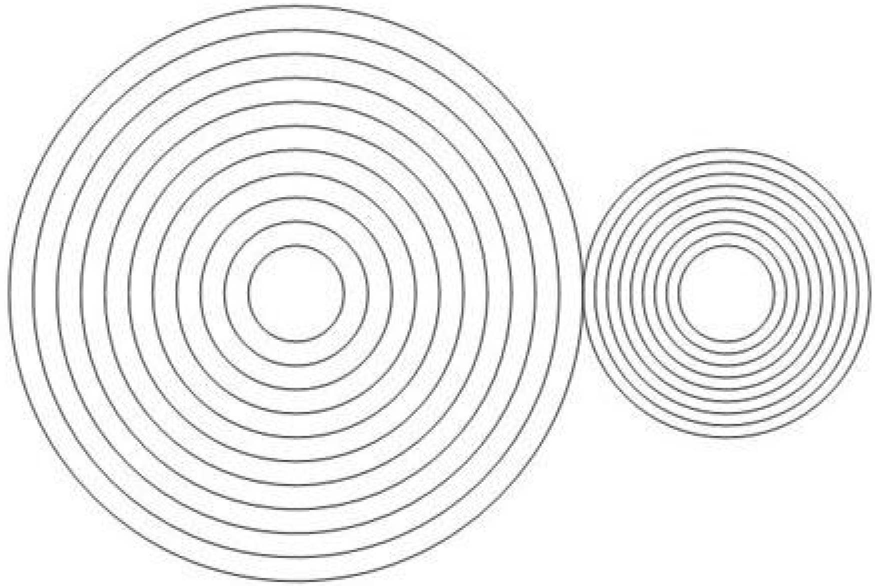


Figure 3: Growing circles that eventually collide

If the two centers are 110 cm apart, then at which moment in time, if any, will the circles first 'touch' or 'collide?'

a) This will happen at $t = \dfrac{441 - \sqrt{881}}{2} \approx 205.66$ seconds.
b) This will happen at $t = 33$ seconds.
c) This happens with the the initial circles, at $t = 0$ seconds
d) This will happen at $t = 4033$ seconds.
e) None of the other answers provided.

25) Consider the function $f(x) = a^x + x^a - b$ where a and b are both positive numbers. If f has a local minimum at $(1, 1)$, then what can we conclude about the values of a and b?

a) $a = e^{-1}$, $b = e$
b) $a = b = e$
c) $a = b = e^{-1}$
d) $a = e$, $b = e^{-1}$
e) None of the other answers provided.