Calculus Exam

University of Houston Math Contest 2025

Problem 1. Evaluate the following limit.

$$\lim_{x \to 0} \frac{\sin(5x)}{3x}$$

(A) $\frac{5}{3}$
(B) $\frac{3}{5}$
(C) 0
(D) 1
(E) does not exist as a real number
(F) none of the above

Problem 2. The variables x and y are related implicitly by

$$xy + e^{y^2} = 5.$$

What is the slope of the tangent line to the graph of this relation at the point $(5 - e, 1)$?

(A) 0
(B) $-\frac{1}{5}$
(C) $\frac{4}{5+e}$
(D) $-\frac{1}{5+e}$
(E) $\frac{1}{5-e}$
(F) none of the above

Problem 3. If the product of two positive real numbers is equal to 7, what is the minimal possible value of the sum of the squares of these two numbers?

(A) $\frac{7}{2}$
(B) 7
(C) 14
(D) 28
(E) 49
(F) none of the above

Problem 4. The variables x and y are related by

$$\tan(y) = x.$$

What is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

(A) $\frac{1}{1+x^2}$

(B) $-\frac{1}{1+x^2}$

(C) $\frac{1}{\sqrt{1-x^2}}$

(D) $-\frac{1}{\sqrt{1-x^2}}$

(E) $\sec^2(x)$

(F) none of the above

Problem 5. A spherical balloon inflates at a rate of 800 cubic meters per second. What is the rate of change of the radius of the balloon at the instant the balloon has volume $\frac{500\pi}{3}$ cubic meters?

(A) $\sqrt[3]{800}$ m/s

(B) $\frac{8}{\pi}$ m/s

(C) 8π m/s

(D) -4 m/s

(E) $\frac{4}{\pi}$ m/s

(F) none of the above

Problem 6. Evaluate the following derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\cos(x)} \frac{1}{2+t^3}\,\mathrm{d}t.$$

(A) $\frac{1}{2+\cos^3(x)}$

(B) $\frac{\tan(x)}{2+\cos^3(x)}$

(C) $\int_0^{\cos(x)} -\frac{3t^2}{(2+t^3)^2}\,\mathrm{d}t$

(D) $-\frac{\sin(x)}{2+\cos^3(x)}$

(E) $\frac{1}{2+\cos^3(x)} - \frac{1}{2}$

(F) none of the above

Problem 7. Let f and g be differentiable functions with domain $(-\infty, \infty)$. Suppose we know the following.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	7	8	-2	4
7	11	-3	5	9

Define the function h with domain $(-\infty, \infty)$ by $h(x) = g(f(x))$. What is $h'(3)$?

(A) 32
(B) -27
(C) 35
(D) -12
(E) 72
(F) none of the above

Problem 8. Define the function f on domain $(0, \infty)$ by $f(x) = x^x$. What is $f'(x)$?

(A) $x^x \ln(x)$
(B) $x^x(1 + \ln(x))$
(C) x^x
(D) $x^x(2 + \ln(x))$
(E) $e \cdot x^x$
(F) none of the above

Problem 9. Define the function f on domain $(-\infty, \infty)$ by

$$f(x) = \int_0^x e^{-t^2}\,\mathrm{d}t.$$

Evaluate the antiderivative

$$\int x f'(x)\,\mathrm{d}x.$$

(A) $2e^{-x^2} + C$
(B) $-\left(\frac{1}{2}\right) e^{-x} + C$
(C) $-\left(\frac{1}{2}\right) e^{-x^2} + C$
(D) $e^{x^2} + C$
(E) $e^{-x^3} + C$
(F) none of the above

Problem 10. Suppose f is a differentiable function with domain $(-\infty, \infty)$ and derivative

$$f'(x) = \frac{7x^3}{3} - \frac{x^4}{4}.$$

For what values of c, if any, does the graph of f have an inflection point at $(c, f(c))$?

(A) 0
(B) 7
(C) 0 and 7
(D) 0 and $\frac{28}{3}$
(E) The graph of f has no inflection points.
(F) none of the above

Problem 11. The concentration $N(t)$ of bacteria at time t obeys the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = KN\left(1 - \frac{N}{L}\right),$$

where K and L are positive constants. Suppose that $N(0) = \frac{L}{4}$. What is

$$\lim_{t \to \infty} N(t)?$$

(A) 0
(B) $\frac{L}{4}$
(C) $\frac{L}{2}$
(D) $\frac{3L}{4}$
(E) L
(F) none of the above

Problem 12. Evaluate the following limit.

$$\lim_{x \to \infty} \frac{x^{2025}}{e^x}$$

(A) 0
(B) 1
(C) 2025
(D) $(2025)(2024)(2023)\cdots(2)(1)$
(E) does not exist as a real number
(F) none of the above

Problem 13. Define the function f on domain $(0, \infty)$ by

$$f(x) = \begin{cases} x^2, & \text{if } x \leqslant e, \\ K + M\ln(x), & \text{if } x > e, \end{cases}$$

where K and M are constants. Find the values of K and M such that f is differentiable at $x = e$.

(A) $K = e^2$, $M = 0$
(B) $K = e^2 - 1$, $M = 1$
(C) $K = e$, $M = e$
(D) $K = 0$, $M = e^2$
(E) $K = -e^2$, $M = 2e^2$
(F) none of the above

Problem 14. Suppose f is a continuous function with domain $(-\infty, \infty)$ such that $f(-x) = -f(x)$ for every real x. Which one of the following could be the value of the integral

$$\int_{-2025}^{2025} \sin(x^2) f(x)\, dx?$$

(A) 1
(B) 2025
(C) π
(D) 2
(E) -1
(F) none of the above

Problem 15. Evaluate the following limit.

$$\lim_{n\to\infty} \left(1 + \frac{2025}{n}\right)^{3n}$$

Here n takes integer values.

(A) 1
(B) 0
(C) e
(D) e^{2028}
(E) e^{6075}
(F) none of the above

Problem 16. What is the equation of the curve that passes through the point $(0,4)$ and such that the slope of the tangent line to the curve at any point (x,y) on the curve is $\frac{x}{y}$?

(A) $y = x + 4$
(B) $y = x^2 + 4$
(C) $y = \sqrt{x^2 + 16}$
(D) $y = \sqrt{x^4 + 16}$
(E) $y = x^4 + 4$
(F) none of the above

Problem 17. Suppose f is a continuous function with domain $[0,6]$ such that

$$\int_1^2 f(x)\,\mathrm{d}x = 5, \quad \int_0^3 f(x)\,\mathrm{d}x = 9, \quad \text{and} \quad \int_0^6 f(x)\,\mathrm{d}x = 17.$$

Evaluate

$$\int_1^2 f(3x)\,\mathrm{d}x.$$

(A) $\frac{8}{3}$
(B) $\frac{5}{3}$
(C) 15
(D) 5
(E) $\frac{17}{3}$
(F) none of the above

Problem 18. Let R be the region in the plane bounded by $x = 1$, $x = L$, $y = 0$, and $y = x^3$, where $L > 1$ is an unknown. Suppose that when we rotate the region R around the x-axis, we obtain a solid with volume 7π. What is the value of L?

(A) $\sqrt[4]{29}$
(B) $\sqrt[4]{28}$
(C) 5π
(D) $\sqrt[7]{50}$
(E) $\sqrt[7]{49}$
(F) none of the above

Problem 19. A spring that obeys Hooke's law has a natural (rest) length of 3 meters. It requires 100 Newtons (N) of force to hold the spring at a length of 1 meter. How much work is done when compressing the spring from its rest length to a length of 2 meters?

(A) 25 N
(B) 50 N
(C) 100 N
(D) 200 N
(E) 400 N
(F) none of the above

Problem 20. Suppose f is a differentiable function with domain $(-\infty, \infty)$ such that $f(2) = 4$ and $|f'(x)| \leqslant 7$ for every x in the interval $(2, L)$, where $L > 2$ is a parameter. What is the smallest value of L for which it is possible to have $f(L) \leqslant -17$?

(A) $L = \frac{9}{2}$
(B) $L = 4$
(C) $L = 5$
(D) $L = 6$
(E) $L = 23$
(F) none of the above

Problem 21. Evaluate

$$\lim_{n\to\infty} \frac{\pi}{2n} \sum_{i=1}^{n} \cos\left(\frac{i\pi}{2n}\right).$$

Hint: Interpret the expression as a definite integral.

(A) π
(B) $\frac{\pi}{2}$
(C) 0
(D) 1
(E) 2π
(F) none of the above

Problem 22. Let f be a function with domain $(-\infty,\infty)$. Suppose there exists a positive integer n such that f is n-times differentiable on $(-\infty,\infty)$ and such that the n^{th} derivative of f, $f^{(n)}$, satisfies $f^{(n)}(x)=0$ for every real x. Which one of the following statements must be true about f?

(A) f is a polynomial of degree at most $n-1$.

(B) There exist constants c and d such that $f(x)=cx+d$ for every x in $(-\infty,\infty)$.

(C) f is a constant function.

(D) f is concave up on $(0,\infty)$.

(E) For every $x\geqslant 0$, we have $-e^x\leqslant f(x)\leqslant e^x$.

(F) none of the above

Problem 23. Let a and b be positive real numbers with $a<b$. Let R be the region in the plane bounded by $x=a$, $x=b$, $y=e^{-x}$, and $y=e^x$. Which one of the following integrals represents the volume of the solid that is produced when R is rotated around the line $y=-5$?

(A) $\pi\int_a^b(e^x-e^{-x})^2\,\mathrm{d}x$

(B) $\pi\int_a^b(e^{-x}+5)^2-(e^x+5)^2\,\mathrm{d}x$

(C) $\pi\int_a^b(e^{2x}-e^{-2x})\,\mathrm{d}x$

(D) $\pi\int_a^b(e^x+5)^2-(e^{-x}+5)^2\,\mathrm{d}x$

(E) $\pi\int_a^b(e^x-5)^2-(e^{-x}-5)^2\,\mathrm{d}x$

(F) none of the above

Problem 24. Define the function f with domain $(-\infty,\infty)$ by

$$f(x)=\begin{cases}0, & \text{if } x=0,\\ x^2\sin\left(\frac{1}{x^3}\right), & \text{if } x\neq 0.\end{cases}$$

Which one of the following statements is true about f?

(A) f is not continuous at $x=0$.

(B) f is continuous at $x=0$ but f is not differentiable at $x=0$.

(C) f is differentiable at $x=0$ and $f'(0)=0$.

(D) f is differentiable at $x=0$ and $f'(0)=\frac{2}{3}$.

(E) f is differentiable at $x=0$ and $f'(0)=\frac{3}{2}$.

(F) none of the above

Problem 25. Suppose f is a function that is differentiable on $(-\infty, \infty)$. Suppose that the equation $f'(c) = 0$ has exactly 3 solutions, $c_1 < c_2 < c_3$. Which one of the following statements must be true about the equation $f(x) = 0$?

(A) It has at least 1 solution.

(B) It has at least 2 solutions.

(C) It has at most 3 solutions.

(D) It has exactly 4 solutions.

(E) It has at most 4 solutions.

(F) none of the above

Problem 26. A string of length L is cut into two pieces, the first of length x and the second of length $L - x$. The first piece is shaped into a circle and the second piece is shaped into a square. What value of x minimizes the sum of the areas of the circle and square?

(A) $\frac{L}{2}$

(B) $\frac{L}{\pi}$

(C) $\frac{\pi L}{4+\pi}$

(D) $\frac{4\pi L}{1+4\pi}$

(E) $\frac{3L}{\pi}$

(F) none of the above

Problem 27. A lighthouse stands upon a small island at distance 4 km from the nearest point Q to the straight shoreline. Light from the lighthouse makes 3 complete revolutions per minute. How fast is the light from the lighthouse moving along the shoreline when it is 2 km away from the point Q?

(A) 10π km/min

(B) 20π km/min

(C) 30π km/min

(D) 40π km/min

(E) 50π km/min

(F) none of the above

Problem 28. The position of a particle on the x-axis is given for time $t > 0$ by

$$x(t) = \ln \left[e^{-t}(t^2 + 2t) \right].$$

For what values of $t > 0$ is the particle moving to the right?

(A) $(0, \sqrt{2})$
(B) $(\sqrt{2}, \infty)$
(C) $(0, \infty)$
(D) $(0, 2)$
(E) $(2, \infty)$
(F) none of the above

Problem 29. Suppose f is a differentiable function with domain $(-\infty, \infty)$ such that

$$f(x + h) = f(x) + f(h) + x^2h + xh^2$$

for every real x and real h. If

$$\lim_{c \to 0} \frac{f(c)}{c} = 5,$$

then what is the function f'?

(A) $f'(x) = 5 - x^2$
(B) $f'(x) = 5 + x^2$
(C) $f'(x) = x^2 - 5$
(D) $f'(x) = 1$
(E) $f'(x) = 5$
(F) none of the above

Problem 30. Let f be a continuous function with domain $[0, 2025]$ such that

$$f(x)f(2025 - x) = 1$$

for every x in $[0, 2025]$. Evaluate

$$\int_0^{2025} \frac{dx}{1 + f(x)}.$$

Hint: Make use of the substitution $u = 2025 - x$.

(A) 2025
(B) 4050
(C) $\frac{2025}{2}$
(D) $\frac{2025}{4}$
(E) 8100
(F) none of the above

Problem 31. Let $b > 1$ be an unknown real number. Find the unique value of $b > 1$ for which the equation

$$b^x = \log_b(x)$$

has exactly one solution. Hint: The functions b^x and $\log_b(x)$ are inverses, so where must any point at which these two curves intersect lie?

(A) e

(B) e^2

(C) e^{-e}

(D) $e^{1/e}$

(E) 2

(F) none of the above

Problem 32. Suppose f is a differentiable function defined on $(-\infty, \infty)$. The real number p is called a *fixed point* of f if $f(p) = p$. If $f'(x) \neq 1$ for every real x, then how many fixed points can f have at most?

(A) 0

(B) 1

(C) 2

(D) 3

(E) There is no upper bound on the number of fixed points that f can have.

(F) none of the above