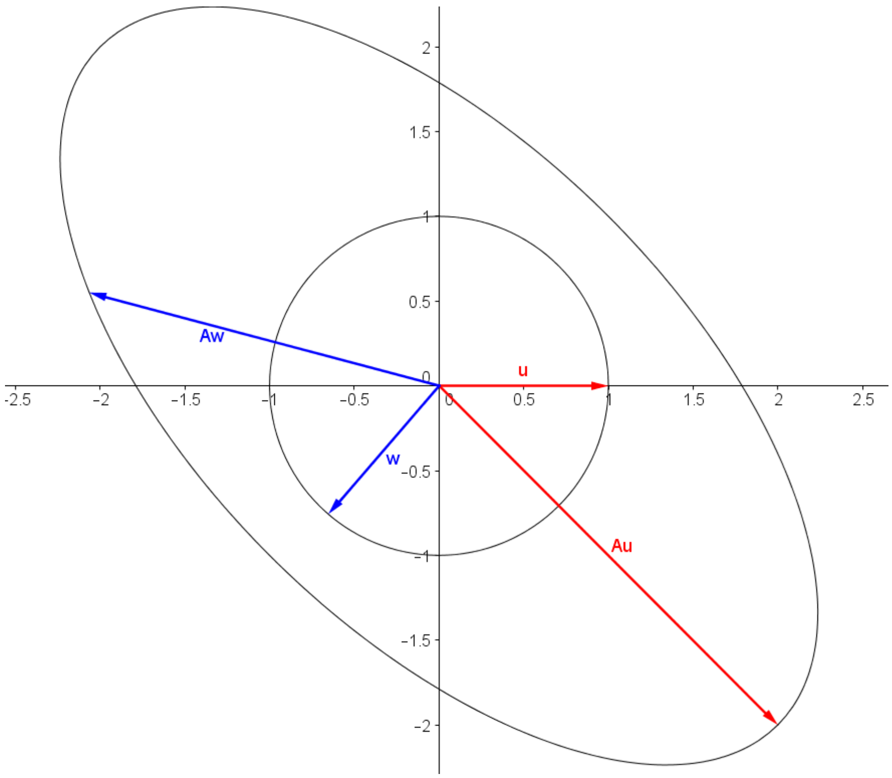
# University of Houston – Math Contest
# Project Problem, 2013

## *Matrices and Matrix Multiplication*




**School Name: _______________________________________**

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| **Team Members** | |
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The Project is due at 8:30am (during onsite registration) on the day of the contest. **Each project will be judged for precision, presentation, and creativity.** If you are attending the contest, then place the team submission in a folder with this page as the cover page, a table of contents, and the solutions. Otherwise, email a pdf file to jmorgan@math.uh.edu. Solve as many of the problems as possible.

**Remark:** Whenever a matrix or vector is referred to below, it is assumed that the entries in the matrix or vector are real numbers.

1. Learn about row vectors, column vectors, and matrices. Explain how to add and subtract vectors and matrices, and multiply vectors and matrices by scalars (real numbers). Make sure you give clear instructions concerning any size requirements for the vectors and matrices.
2. Learn how to multiply matrices by column vectors, and matrices by matrices. Then explain the process. As in problem 1, make sure you give clear instructions concerning any size requirements for the matrices and vectors.
3. The picture on the cover of this problem set was created as follows. First, a matrix $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ was created, and the circle was parameterized by $(\cos(t), \sin(t))$ for $0 \le t \le 2\pi$.
   Then a new parameterization was created by first performing the multiplication
   $$\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = \begin{pmatrix} 2\cos(t) + \sin(t) \\ -2\cos(t) + \sin(t) \end{pmatrix}$$
   and then plotting the curve parameterized by $(2\cos(t) + \sin(t), -2\cos(t) + \sin(t))$ for $0 \le t \le 2\pi$ (notice how the parameterization results from the multiplication shown above). Show that the curve generated truly is an ellipse. That is, show that the curve generated is given by an expression of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ where $a, b, c, d, e,$ and $f$ are real numbers, and $b^2 - 4ac < 0$.
4. Suppose a parameterization for a curve is created in a manner similar to the previous problem, using an arbitrary 2 by 2 matrix of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma$ and $\delta$ are real numbers. Show that the resulting curve is an ellipse if and only if $\alpha\delta - \beta\gamma \ne 0$, and that the area of the ellipse is given by $|\alpha\delta - \beta\gamma|\pi$.
5. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is the so-called 2 by 2 identity matrix. Prove that $AI = IA = A$ for every 2 by 2 matrix $A$.
6. For a given 2 by 2 matrix $A$ define $C_A$ to be the set of all 2 by 2 matrices $B$ so that $AB = BA$. Find a 2 by 2 matrix $B$ which is not in $C_A$ for $A = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$. Then give an explicit description of the set of all matrices in $C_A$.
7. Let $A$ be a 2 by 2 matrix. Prove that if $C, D \in C_A$, then $\alpha C$, $C + D$ and $CD$ are in $C_A$ for every real number $\alpha$.
8. Let $A$ be a 2 by 2 matrix. Prove that if $\alpha$ and $\beta$ are real numbers, then $\alpha I + \beta A \in C_A$.

9. State and prove a necessary and sufficient condition for a 2 by 2 matrix $A$ to satisfy $C_A = \{\alpha I + \beta A \mid \alpha, \beta \in R\}$.

10. Suppose $n$ is a positive integer, and let $M_n$ denote the set of $n$ by $n$ matrices. Suppose $f$ is a function whose domain is $M_n$, and whose range is the set of real numbers. Furthermore, suppose $f$ satisfies the properties $f(A+B) = f(A) + f(B)$, $f(\alpha A) = \alpha f(A)$ and $f(AB) = f(BA)$ whenever $A$ and $B$ are in $M_n$, and $\alpha$ is a real number.

    a. Suppose $n = 2$ and $g$ satisfies the properties that $f$ satisfies above. In addition, suppose $g\left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\right) = 1$. State and prove a formula for $g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$.

    b. Suppose $n$ is an arbitrary (but given) positive integer, and $g$ satisfies the properties that $f$ satisfies above. Show that if there is a matrix $B$ in $M_n$ for which $g(B) \neq 0$, then there is a real number $k$ so that $f(A) = k\,g(A)$ for every matrix $A$ in $M_n$.